Lesson 8 Fourier series converge! ... but how fast?

Bessel's ineq: φ_n continuous on $[-\pi,\pi]$, orthonormal.

$f(x)$ piecewise continuous on $[-\pi,\pi]$. Define $c_n = \langle f, \varphi_n \rangle$.

[Note: Don't know anything about whether $f = \sum_1^\infty c_n \varphi_n$.]

Then $\sum_1^\infty |c_n|^2 \le \int_{-\pi}^{\pi} |f|^2\,dx$. So $c_n \to 0$ as $n \to \infty$.

Prime example: $\varphi_1(x) = \frac{1}{\sqrt{2\pi}}$, $\varphi_{2n}(x) = \frac{\sin nx}{\sqrt{\pi}}$, $\varphi_{2n+1}(x) = \frac{\cos nx}{\sqrt{\pi}}$

Cor: Fourier coeff of a piecewise cont fcn on $[-\pi,\pi]$ tend to zero as $n\to\infty$. [Real or complex versions]

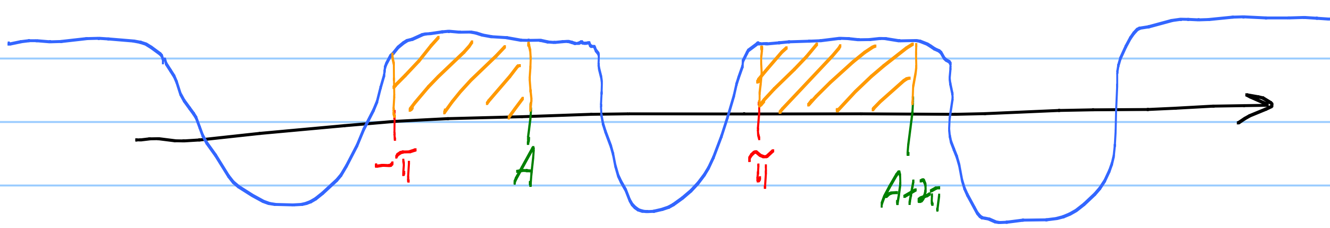
Today: Assume f is C^1-smooth on $\mathbb{R}$ and 2π-periodic.

Thm: Fourier series $\underbrace{a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) = \sum_{n=-N}^{N} c_n e^{inx}}_{S_N(x)}$ converge to $f(x)$ at each $x \in \mathbb{R}$.

Lemma f 2π-periodic, then

$$\int_{-\pi}^{\pi} f\,dx = \int_{A}^{A+2\pi} f\,dx$$

Pf:



Pf of thm: $S_N(x) = \sum_{n=-N}^{N} c_n e^{inx}$

$$= \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt \right) e^{inx}$$

$$= \int_{-\pi}^{\pi} \underbrace{\left(\frac{1}{2\pi} \sum_{n=-N}^{N} e^{in(x-t)} \right)}_{!} f(t)\, dt$$

Dirichlet kernel : $D_N(t) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{int}$ (2π new!)

Fantastic formula : $\boxed{S_N(x) = \int_{-\pi}^{\pi} D_N(x-t)\, f(t)\, dt}$

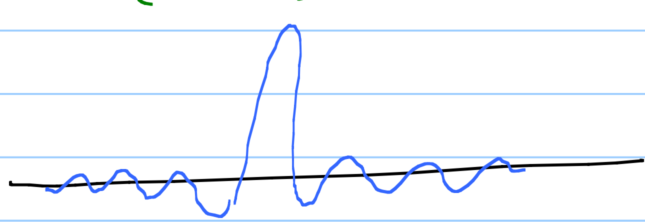
Facts: 1) $\int_{-\pi}^{\pi} D_N(t)\, dt = 1$

2) $D_N(t) = \frac{1}{2\pi} \frac{\sin\left(N+\frac{1}{2}\right)t}{\sin t/2}$

and $D_N(0) = \frac{1}{2\pi}(2N+1)$ (L'H)

3) $D_N(t)$ is 2π-periodic

(because each $e^{\pm int}$ is too.)

4) 

Pf of (1): $\displaystyle\int_{-\pi}^{\pi} D_N(t)\,dt = \underbrace{\int_{-\pi}^{\pi} \tfrac{1}{2\pi}\cdot 1}_{\substack{2\pi\cdot\frac{1}{2\pi}\\ =1}} + \sum_{\substack{|n|\le N\\ n\neq 0}} \underbrace{\int_{-\pi}^{\pi} e^{int}\,dt}_{=0}$

because $e^{int} = \cos nt + i\sin nt$

Bad things about D_N: 1) $D_N(t)$ is not ≥ 0.

2) $\displaystyle\int_{-\pi}^{\pi} |D_N(t)|\,dt \longrightarrow \infty$ as $N \to \infty$.

Pf cont'd: $\displaystyle S_N(x) = \int_{-\pi}^{\pi} D_N(\underbrace{x-t}_{\tilde{\tau}})\,f(t)\,dt$

$\tau = x - t \leftarrow t = x - \tau$

$d\tau = -dt$

When $t = -\pi$: $\tau = x + \pi$

$t = \pi$: $\tau = x - \pi$

$$= \int_{\tau = x+\pi}^{x-\pi} D_N(\tau)\, f(x-\tau)\,(-d\tau)$$

$$= \int_{x-\pi}^{x+\pi} D_N(\tau)\, f(x-\tau)\,d\tau$$

Lemma $\Longrightarrow$ $\displaystyle = \int_{-\pi}^{\pi} D_N(t)\, f(x-t)\,dt$

changed τ to t and shifted $\int$

Hmmm: $\displaystyle f(x) = 1 \cdot f(x) = \int_{-\pi}^{\pi} D_N(t)\,dt \cdot f(x)$

$$= \int_{-\pi}^{\pi} D_N(t)\, f(x)\,dt$$

Finally $$\boxed{S_N(x) - f(x) = \int_{-\pi}^{\pi} D_N(t)\big(f(x-t) - f(x)\big)\,dt}$$

Not hard to see this $\to 0$ as $N \to \infty$.

$$\int_{-\pi}^{\pi} \frac{1}{2\pi}\,\frac{\sin(N+\frac{1}{2})t}{\sin t/2}\,\big(\underline{f(x-t) - f(x)}\big)\,dt$$

$$\sin(N+\tfrac{1}{2})t = \underline{\sin Nt}\,\cos\tfrac{t}{2} + \underline{\cos Nt}\,\sin\tfrac{t}{2}$$

$$S_N(x) - f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin Nt\,\underbrace{\left[\frac{1}{2}\cos\frac{t}{2}\,\underbrace{\frac{f(x-t)-f(x)}{\sin\frac{t}{2}}}_{g(t)}\right]}_{h(t)}dt$$

$$+\,\frac{1}{\pi}\int_{-\pi}^{\pi} \cos Nt\,\underbrace{\left[\frac{1}{2}\sin\frac{t}{2}\,\frac{f(x-t)-f(x)}{\sin\frac{t}{2}}\right]}_{g(t)}dt$$

Aha! These are the N-th Fourier coeff for <u>nice</u> fcns

<u>Why</u>: L'H: $\displaystyle\lim_{t\to 0}\frac{f(x-t)-f(x)}{\sin\frac{t}{2}} = \lim_{t\to 0}\frac{-f'(x-t)}{\frac{1}{2}\cos\frac{t}{2}} = -2f'(x)$

Bessel's $\Rightarrow$ these $\to 0$ as $N \to \infty$. Done!

But how fast? Feeling: the <u>smoother</u> the fcn, the faster the convergence.

Next time: f is C^2-smooth, 2π-periodic,

then $\left|S_N(x) - f(x)\right| < \frac{M}{N}$ for some

constant $M > 0$ and all x.

... C^3-smooth ... $< \frac{M}{N^2}$

... C^k-smooth ... $< \frac{M}{N^{k-1}}$

...

History: Bessel's $\Rightarrow S_N \to f$ in L^2-norm!

This is where we discovered that Riemann integral fails us! Need Lebesgue version of integral.