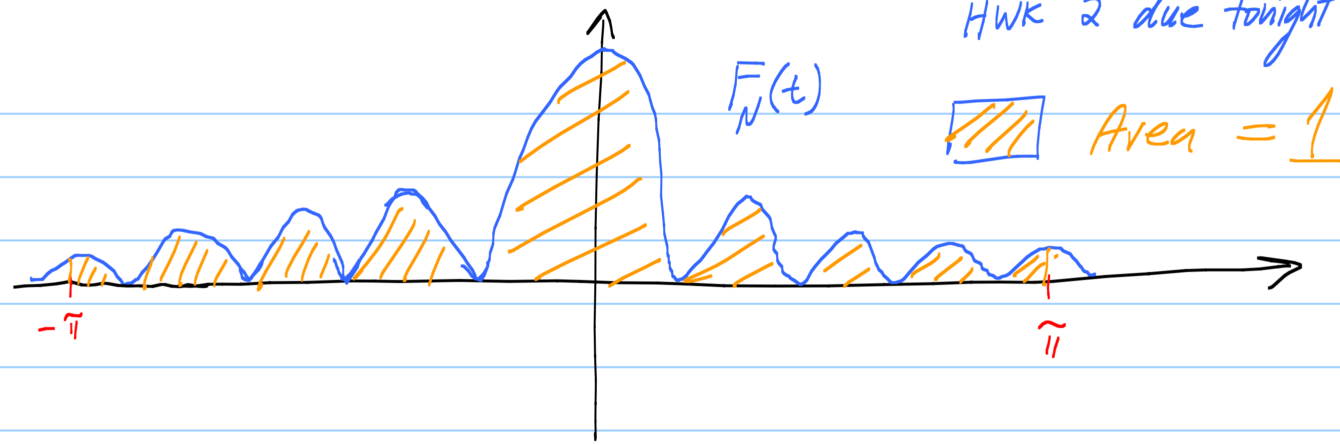
Lesson 10 The Fejér kernel and Fejér's theorem

HWK 2 due tonight 11pm



EX: Stein: A continuous 2π-periodic fcn whose F.S. does not converge at pt x_0.

Big idea: $S_N(x) = \sum_{-N}^{N} c_n e^{inx}$ $\quad c_n = \hat{f}(n)$

Assume f cont., 2π-periodic.

Consider the average up to N of $S_n(x)$:

$$\sigma_N(x) = \frac{S_0(x) + S_1(x) + \cdots + S_{N-1}(x)}{N}$$

$$= \int_{-\pi}^{\pi} \underbrace{\left[\frac{D_0(t) + D_1(t) + \cdots + D_{N-1}(t)}{N}\right]}_{F_N(t)} f(x-t)\, dt$$

$F_N(t) \leftarrow$ Fejér kernel

Geometric series in e^{ix}, e^{-ix}. Use sum formula. Clean up cleverly via algebra. Get

$$F_N(t) = \frac{1}{2\pi N} \frac{\sin^2\left(\frac{Nt}{2}\right)}{\sin^2\left(\frac{t}{2}\right)} \quad !$$

Key properties: 1) $F_N(t) \geq 0$ for all t.

pf. Formula.

2) $\int_{-\pi}^{\pi} F_N(t)\,dt = 1$ pf: $D_n(t)$'s do that

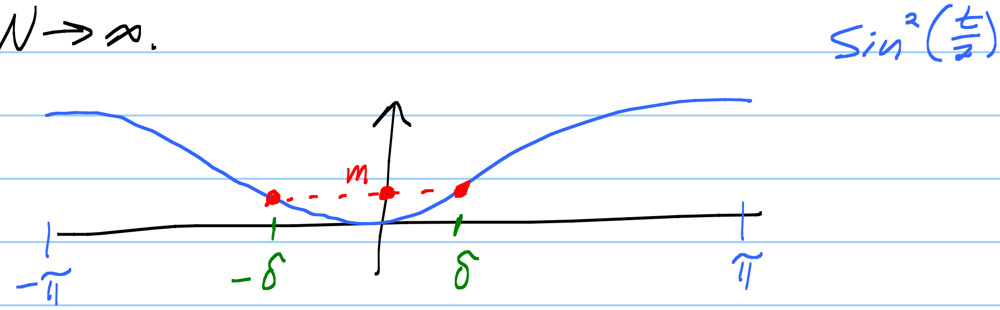
3) $\lim_{t\to 0} F_N(t) = \frac{N}{2\pi} \longrightarrow \infty$ as $N\to\infty$. Pf: L'H

4) <u>But</u>, outside any interval $(-\delta,\delta)$, $0<\delta<\pi$

$F_N(t)\to 0$ <u>uniformly</u> on $[-\pi,-\delta]\cup[\delta,\pi]$

as $N\to\infty$.

<u>Pf</u>:



$$|F_N(t)| = \frac{1}{2\pi N}\left|\frac{\sin^2 N\frac{t}{2}}{\sin^2\frac{t}{2}}\right| \le \frac{1}{2\pi N}\cdot\frac{1}{m} \xrightarrow[N\to\infty]{} 0$$

if $t\notin(-\delta,\delta)$

Fejér kernel is a "good" kernel.

<u>Thm</u>: If f is cont and 2π-periodic, then

$\sigma_N(x)\to f(x)$ <u>uniformly</u> on $\mathbb{R}$.

<u>Pf</u>:

$$\sigma_N(x)-f(x) = \int_{-\pi}^{\pi}F_N(t)f(x-t)\,dt - \underbrace{\int_{-\pi}^{\pi}F_N(t)\,dt}_{=1}\cdot f(x)$$

$$= \int_{-\pi}^{\pi} F_N(t)\,\left[f(x-t) - f(x)\right]\,dt$$

$$= \underbrace{\int_{-\delta}^{\delta} F_N(t)\left[f(x-t)-f(x)\right]dt}_{\substack{I_1 \\ \text{second term small}}} + \underbrace{\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right) F_N(t)\left[f(x-t)-f(x)\right]dt}_{\substack{I_2 \\ \text{first term small}}}$$

Let $\varepsilon > 0$. f cont on $[-\pi,\pi] \Rightarrow f$ unif cont

$\Rightarrow f$ unif cont on $\mathbb{R}$

So $\exists\ \delta > 0$ such that $|f(x-t) - f(x)| < \frac{\varepsilon}{2}$

when $|t| < \delta$ (independent of x!).

$$|I_1| \le \int_{-\delta}^{\delta} \underbrace{|F_N(t)|}_{=F_N(t)}\ \underbrace{|f(x-t)-f(x)|}_{\le \frac{\varepsilon}{2}}\ dt$$

$$\le \frac{\varepsilon}{2}\int_{-\delta}^{\delta} F_N(t)\,dt \le \frac{\varepsilon}{2}\underbrace{\int_{-\pi}^{\pi} F_N(t)\,dt}_{=1}$$

$|I_1| \le \frac{\varepsilon}{2}$ ✓

Note: f cont on $[-\pi,\pi] \Rightarrow f$ bounded there.

Say $|f| \le M$ on $[-\pi,\pi]$. 2π-periodic: $|f| \le M$ on $\mathbb{R}$.

So $|f(x-t)-f(x)| \le |f(x-t)| + |f(x)| \le \underbrace{M+M}_{2M}$

Now $|I_2| = \left| \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} \right) F_N(t)\,[f(x-t)-f(x)]\,dt \right|$

$$\le \underbrace{\left(\max_{\substack{|t|>\delta \\ t\in[-\pi,\pi]}} F_N(t) \right)}_{\text{can make } < \frac{\varepsilon/2}{4\pi M}} \cdot 2M \cdot 2\pi$$

by taking N large enough. $> N_0$

So $|\sigma_N(x) - f(x)| < \underbrace{\frac{\varepsilon}{2} + \frac{\varepsilon}{2}}_{\varepsilon}$

if $N > N_0$.

Fejér's thm 2: Trigonometric polynomials are dense among 2π-periodic cont fcns on $\mathbb{R}$.

Huge consequence: Bessel's ineq for Fourier series is an equality: Parseval's identity.

Why: $$0 \le \underbrace{\left\| f - \sum_{-N}^{N} c_n e^{inx} \right\|^2}_{\|f\|^2 - \sum_{-N}^{N} \frac{|c_n|^2}{2\pi}} \le \underbrace{\left\| f - \underbrace{\sum_{-N}^{N} A_n e^{inx}}_{\text{use } \sigma_N(x) \text{ here}} \right\|^2}_{\to 0 \text{ as } N\to\infty.}$$

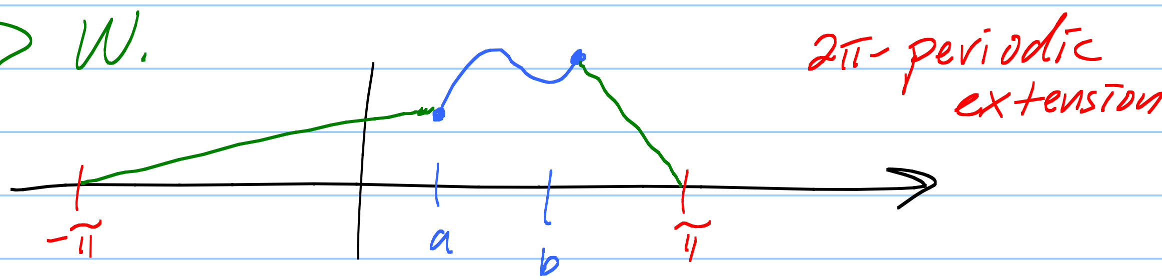
$c_n = \hat{f}(n)$

$$0 \le \|f\|^2 - \sum_{-N}^{N} \frac{|c_n|^2}{2\pi} \le \varepsilon \quad \text{for } N > N_0.$$

Aha! $$\lim_{N\to\infty} \sum_{-N}^{N} |c_n|^2 = 2\pi \|f\|^2$$
$$= 2\pi \int_{-\pi}^{\pi} |f|^2\,dx$$

Fun thing: Weierstraß thm: Poly's are dense among cont fcns on a closed interval.

Fejér's $\Rightarrow$ W.



Get trig poly close to it.

$$e^{inx} = \underbrace{\cos nx}_{\text{power series!}} + i\,\underbrace{\sin nx}$$

Approx power series by partial sums.