# Lesson 11 Solution to the Dirichlet problem on the disc

Formula sheet on Homepage

Parseval's

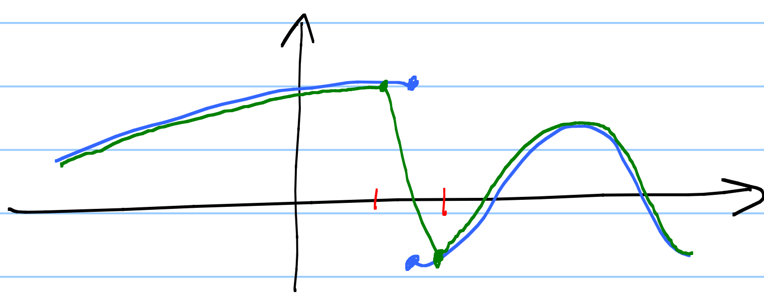
$$0 \le \underbrace{\left\| f - \sum_{-N}^{N} c_n e^{inx} \right\|^2}_{\| f\|^2 - \frac{1}{2\pi}\sum_{-N}^{N}|c_n|^2} \le \underbrace{\left\| f - \underbrace{\sum_{-N}^{N} A_n e^{inx}}_{\sigma_N(x)} \right\|^2}_{\text{can make } < \varepsilon \text{ by Fejér's.}}$$

Last time: Assumed $f$ cont. $2\pi$-periodic

Fejér's $\Rightarrow$ Parseval's $\|f\|^2 = \frac{1}{2\pi}\sum_{-\infty}^{\infty}|c_n|^2$.

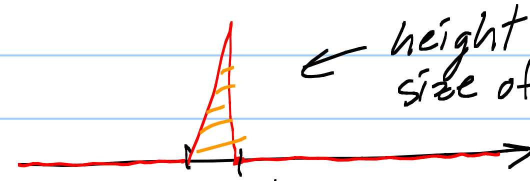
Hmmm: Bessel's holds when $f$ is piecewise cont.

Trick



$\|\text{Green} - \text{Blue}\|^2$ can be made small!

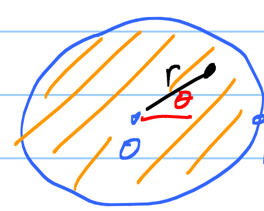
$$\int_{-\pi}^{\pi} |\text{Green} - \text{Blue}|^2 \, dx :$$




small area!

Conclusion: Parseval's holds for piecewise continuous $f$.

Solve heat problem on a metal disc:




$D_1(0)$

Know temp on boundary (circle).

Temp $= u(r,\theta)$. Let it reach "steady state".

Math problem: Need 1) $\Delta u = 0$ on $D_1(0)$

2) $u$ continuous up to circle,

3) $u(1,\theta) = f(\theta) \quad 0 \le \theta < 2\pi$

$$\begin{cases} f(0) = f(2\pi) \\ f'(0) = f'(2\pi) \end{cases}$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \underline{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}$$

Def$^n$: $D_1(0) = \{(x,y) : \sqrt{x^2+y^2} < 1\} = \{(r,\theta) : r < 1\}$

Try $u(r,\theta) = R(r)\Theta(\theta)$

$$\Delta u = 0$$

want

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

divide by $R\Theta$. Multiply by $r^2$.

$$\boxed{\frac{r^2R'' + rR}{R} = -\frac{\Theta''}{\Theta}} = \lambda$$

Both sides must = same constant $\lambda$. [Let $\theta = \frac{\pi}{2}$. Let $r = \frac{1}{2}$.]

Two ODEs:

$$\begin{cases} \Theta'' + \lambda\Theta = 0 \\ r^2R'' + rR' - \lambda R = 0 \end{cases}$$

$\Theta$-problem: $\quad \Theta'' + \lambda \Theta = 0$

periodic boundary conditions: $\begin{cases} \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases}$

$u(r,\theta) = R(r)\,\Theta(\theta) \leftarrow$ no jumps, no kinks at $0 \sim 2\pi$.

$\lambda = 0$: $\quad \Theta''(\theta) \equiv 0$

$\Theta(\theta) = A\theta + B$

$\Theta(0) = B \overset{\text{want}}{=} A \cdot 2\pi + B$

See $\boxed{A = 0.}$

So $\Theta(\theta) \equiv B$

Oh good! $\Theta'(0) = 0 = \Theta'(2\pi)$

Get constant sol$^n$: $\Theta(\theta) \equiv B$. Take $B = 1$.

Write $\boxed{\Theta_0(\theta) = 1}$ for $\lambda = 0$.

$\lambda < 0$: $\quad$ Get $\Theta(\theta) = c_1 e^{k\theta} + c_2 e^{-k\theta}$ $\quad \lambda = -k^2$.

Ouch! Can't satisfy periodic B.C.

Get no sol$^n$s when $\lambda < 0$.

$\lambda > 0$: $\quad$ Get periodic sol$^n$s exactly when $\lambda = n^2$

$n = 1, 2, 3, \ldots$

$$\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$$

Back to $R(r)$ problem: $\quad r^2 R'' + r R' - \lambda R = 0$

Euler eqn!

**Case $\lambda = 0$:** $\quad r^2 R'' + r R' = 0$. $\quad$ Let $V = R'$

(Standard form) (*) $\quad \dfrac{dV}{dr} + \dfrac{1}{r} V = 0 \qquad$ First order linear!

Int. factor $\quad \mu = e^{\int \frac{1}{r}\, dr} = e^{\ln r} = r$

Mult (*) by $\mu$ : $\quad \underbrace{r \dfrac{dV}{dr} + 1 \cdot V}_{\frac{d}{dr}[\underbrace{\mu V}_{rV}]} = 0$

Get $rV = c_1$, a const.

$$V = \frac{c_1}{r}$$

So $\quad R' = \dfrac{c_1}{r}$

$$R = c_1 \underbrace{\ln r}_{\text{nasty term}} + c_2$$

Physical considerations: $\Rightarrow c_1 = 0$.

Get $R(r) \equiv c_2$. Take $c_2 = 1$.

Also get $\quad u_0(r,\theta) = R_0(r)\,\Theta_0(\theta) = \underline{1}$

**Case $\lambda = n^2$:** $\quad r^2 R'' + r R' - n^2 R = 0$

Euler: Try $\quad R(r) = r^p$

$$R'(r) = p\, r^{p-1}$$

$$R''(r) = p(p-1)\, r^{p-2}$$

Plug in $\quad r^2\left(p(p-1)r^{p-2}\right) + r\left(pr^{p-1}\right) - n^2 r^p \overset{\text{want}}{=} 0$

$$\Big[\underbrace{p(p-1) + p - n^2}_{\text{need } = 0}\Big]\, r^p = 0$$

$$p^2 - n^2 = 0$$

$$\boxed{p = \pm n}$$

Get $\quad R(r) = c_1 r^{-n} + c_2 r^n$

nasty! Must have $c_1 = 0$.

$$R_n(r) = r^n$$

$$u_n(r,\theta) = R_n(r)\,\Theta(\theta) = r^n\left(a_n \cos n\theta + b_n \sin n\theta\right)$$