Lesson 12 Solution to the Dirichlet problem and the Poisson kernel

Euler eqns: $Ax^2y'' + Bxy' + Cy = 0$. Try $y = x^p$

$$\underbrace{[Ap(p-1) + Bp + C]}_{\text{need } = 0} x^p = 0$$

Case: Roots r_1, r_2 real and unequal. $y = c_1 x^{r_1} + c_2 x^{r_2}$

Case: r_0 a (real) double root. $y = c_1 x^{r_0} + c_2 (\ln x) x^{r_0}$

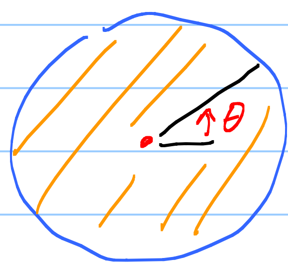
Case: $r = a \pm bi$ complex. For $r = a+bi$:

$$x^{a+bi} = (e^{\ln x})^{a+bi} = e^{a\ln x + i b \ln x} = e^{a\ln x} e^{ib\ln x} = e^{\ln x^a}(\cos(b\ln x) + i\sin(b\ln x)) = x^a\cos(b\ln x) + i x^a \sin(b\ln x)$$

One complex sol^n yields two real sol^{n}s!

Gen^l Sol^n: $y = c_1 x^a \cos(b\ln x) + c_2 x^a \sin(b\ln x)$

Dirichlet problem:



Temp: $u(r,\theta)$. Need 1) u continuous on $\overline{D_1(0)}$

2) $\Delta u = 0$ in $D_1(0)$

3) $u(1,\theta) = f(\theta) \leftarrow$ given

Words: Functions that satisfy the Laplace eqn $\Delta u \equiv 0$ are called harmonic.

$D_1(0)$ = open unit disc $= \{(r,\theta) : 0 \le r < 1\}$.

$\overline{D_1(0)}$ = closure of the disc $= \{(r,\theta) : 0 \le r \le 1\}$.

Last time: did separation variables. Got sol^{n}s

$$u_0(r,\theta) \equiv 1$$

$$u_n(r,\theta) = r^n(a_n \cos n\theta + b_n \sin n\theta)$$

Try $$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta)$$

Finally, we want $u(1,\theta) = f(\theta) \leftarrow$ given

Need $$u(1,\theta) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) = f(\theta)$$ ← want

Aha! Fourier series!

$$\begin{cases} a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta & \leftarrow \text{ave!} \\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \cos n\theta\, d\theta \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \sin n\theta\, d\theta \end{cases}$$

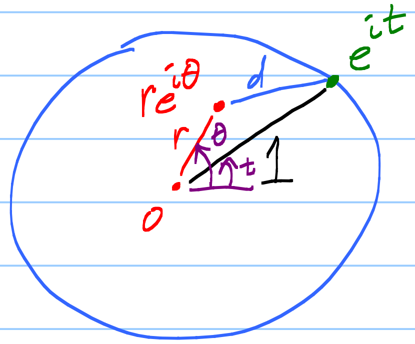
Future: Plug in $r=0$ in sol^n. Get $u(0,\theta) \equiv a_0 =$ ave of temp on boundary!

"Averaging property of harmonic fcns"

Fun thing to do: write out formulas for Fourier series sol^n. Switch to complex notation formula. See geometric series that are easily summed. Get

Poisson formula $$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)} f(t)\, dt$$

Poisson kernel: $$P(r,\theta,t)=\frac{1}{2\pi}\,\frac{1-r^2}{\underbrace{1+r^2-2r\cos(\theta-t)}_{d^2}}$$



See d from Law of Cosines

$$d=\left|re^{i\theta}-e^{it}\right|$$

$$P(r,\theta,t)=\frac{1}{2\pi}\,\frac{1-r^2}{\left|re^{i\theta}-e^{it}\right|^2}$$

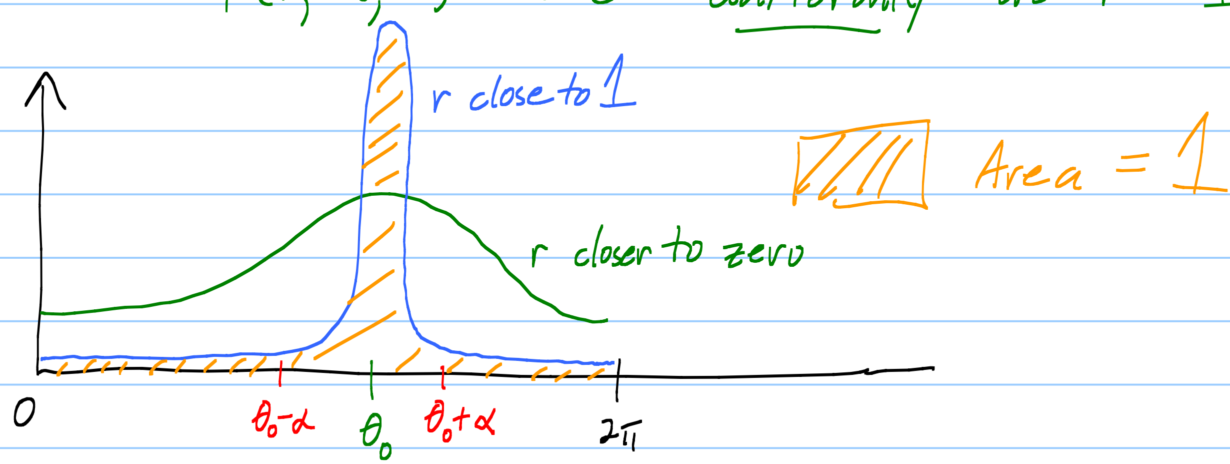
Poisson kernel is a "good" kernel!

1) $P\geq 0$

2) $\displaystyle\int_{-\pi}^{\pi}P(r,\theta,t)\,dt=1 \qquad (r<1)$

3) For t <u>outside</u> $(\theta_0-\alpha,\theta_0+\alpha)$,

$P(r,\theta_0,t)\to 0$ <u>uniformly</u> as $r\to 1$.



4) $\Delta_{(r,\theta)}P(r,\theta,t)\equiv 0$

1–4 will be easy to show.

Cooking up $P(r,\theta,t)$: $u(r,\theta) = \underset{\uparrow\, c_0}{a_0} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$

$$= \lim_{N\to\infty} \left[c_0 + \sum_{n=-N}^{-1} r^{|n|} c_{-n} e^{-in\theta} + \sum_{n=1}^{N} r^n c_n e^{in\theta} \right]$$

$$= \lim_{N\to\infty} \sum_{-N}^{N} r^{|n|} c_n e^{in\theta}$$

$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-int}\, dt$

$$= \lim_{N\to\infty} \int_{-\pi}^{\pi} \Bigg(\underbrace{\frac{1}{2\pi} \sum_{n=-N}^{N} r^{|n|} e^{in(\theta - t)}}_{\substack{\text{two geometric series!} \\ n<0 \\ n\geq 0}} \Bigg) f(t)\, dt$$

Details next time.

Heat eqn: $\frac{\partial u}{\partial t} = c^2 \Delta u$

Steady state: Let $t \to \infty$. Then $\frac{\partial u}{\partial t} \to 0$.

Get Laplace eqn. $\Delta u \equiv 0$.