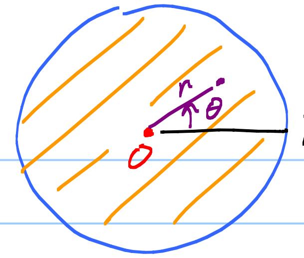
Lesson 13 The Poisson kernel

u cont. on $\overline{D_1(0)}$,
$\Delta u \equiv 0$ in $D_1(0)$,
$u(1,\theta) = f(\theta) \leftarrow$ given



Got $$u(r,\theta) = \underbrace{a_0}_{= c_0} + \sum_{n=1}^{\infty} \underbrace{r^n \Big(a_n \underbrace{\cos n\theta}_{\frac{e^{in\theta}+e^{-in\theta}}{2}} + b_n \underbrace{\sin n\theta}_{\frac{e^{in\theta}-e^{-in\theta}}{2i}}\Big)}_{r^n \left[\underbrace{\frac{(a_n - i b_n)}{2}}_{c_n} e^{in\theta} + \underbrace{\frac{(a_n + i b_n)}{2}}_{c_{-n}} e^{-in\theta} \right]}$$

a's, b's = F. coeff for $f(\theta)$

Partial sum: $$c_0 r^0 + \sum_{-N}^{-1} r^{|n|} c_n e^{in\theta} + \sum_{1}^{N} r^n c_n e^{in\theta}$$

$$= \sum_{-N}^{N} c_n r^{|n|} e^{in\theta}$$

$$= \sum_{-N}^{N} r^{|n|} \underbrace{\left(\frac{1}{2\pi} \int_0^{2\pi} f(t)\, e^{-int}\, dt \right)}_{c_n} e^{in\theta}$$

$$= \int_0^{2\pi} \underbrace{\left(\frac{1}{2\pi} \sum_{-N}^{N} r^{|n|} e^{in(\theta - t)} \right)}_{\frac{1}{2\pi}\left(\sum_{0}^{N} + \sum_{-N}^{-1} \right)} f(t)\, dt$$

Def^n $$\boxed{\begin{aligned} w &= e^{i(\theta - t)} \\ \bar{w} &= e^{-i(\theta - t)} \end{aligned}}$$

2

$$\sum_0^N = 1 + rw + r^2w^2 + \cdots + r^N w^N = \frac{1-(rw)^{N+1}}{1-rw}$$
$$= 1 + (rw) + (rw)^2 + \cdots + (rw)^N =$$

$$\sum_{-1}^{-N} = r\bar{w} + r^2\bar{w}^2 + \cdots + r^N\bar{w}^N$$
$$= (r\bar{w})\left[1 + (r\bar{w}) + \cdots + (r\bar{w})^{N-1}\right] = (r\bar{w})\cdot\frac{1-(r\bar{w})^N}{1-r\bar{w}}$$

Now let $N \to \infty$. Aha! $|(rw)^m| = r^m \underbrace{|w|^m}_{=1} = r^m \to 0$ as $m \to \infty$ when $0 \le r < 1$

So $\displaystyle\sum_{-N}^{N} \to \underbrace{\frac{1}{1-rw} + \frac{r\bar{w}}{1-r\bar{w}}}$ as $N \to \infty$, $(r<1)$

$$= \frac{1 - r\bar{w} + r\bar{w} - r^2 w\bar{w}}{(1-rw)\underbrace{(1-r\bar{w})}_{\text{conj of } 1-rw}} \quad w\bar{w} = |w|^2 = 1$$

$$= \frac{1-r^2}{|1-rw|^2}$$

$$\boxed{z\cdot\bar{z} = |z|^2} \quad \boxed{|zw| = |z||w|}$$

$$= \frac{1-r^2}{|1-re^{i(\theta-t)}|^2} \cdot \underbrace{\frac{1}{|e^{it}|^2}}_{=1}$$

$$= \frac{1-r^2}{|e^{it}-re^{i\theta}|^2} \qquad \text{Done!}$$

Poisson kernel: $\displaystyle P(r,\theta,t) = \frac{1}{2\pi}\,\frac{1-r^2}{|re^{i\theta}-e^{it}|^2} \quad (r<1)$

$$= \frac{1}{2\pi} \cdot \frac{1-r^2}{1+r^2-2r\cos(\theta-t)}$$

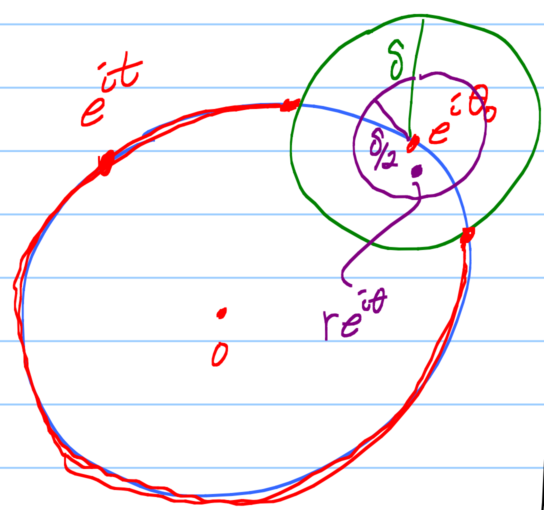
Key properties

1) $P(r,\theta,t) > 0 \quad (0 \le r < 1)$

pf: formula

2) $\int_0^{2\pi} P(r,\theta,t)\,dt = 1$ for $(0 \le r < 1)$

why: $$\int_0^{2\pi} \frac{1}{2\pi}\sum_{-N}^{N} \; dt = \underbrace{\int_0^{2\pi} \frac{1}{2\pi}\cdot 1\,dt}_{=1} + \sum_{n\neq 0} \underbrace{\int (\;\;) e^{int}\,dt}_{=0}$$

3) 

$I_\delta = \{t : e^{it} \notin D_\delta(e^{i\theta_0})\}$

$P(r,\theta,t) \to 0$ unif for $t \in I_\delta$ as $r \nearrow 1$ when $re^{i\theta} \in D_{\delta/2}(e^{i\theta_0})$

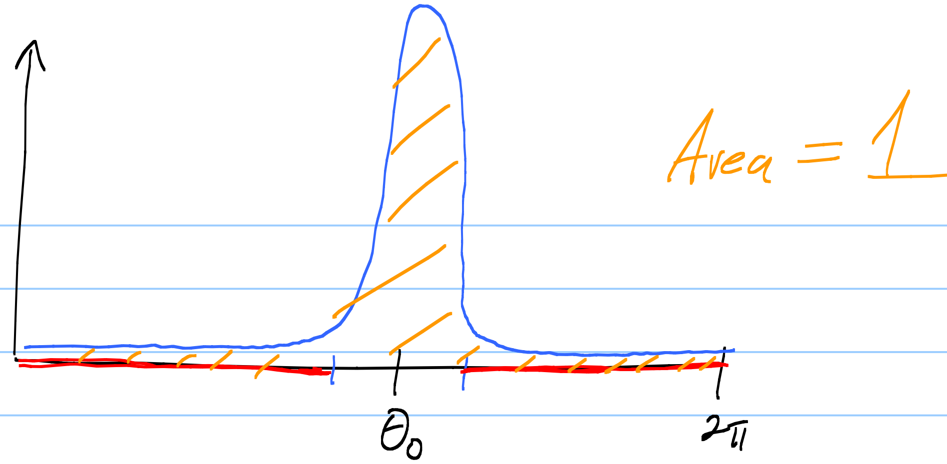
Pf: $$P(r,\theta,t) = \frac{1}{2\pi}\frac{1-r^2}{|re^{i\theta}-e^{it}|^2}$$

$|re^{i\theta} - e^{it}| > \frac{\delta}{2}$ from picture

$$= \frac{1}{2\pi}\frac{(1-r)\overbrace{(1+r)}^{<2}}{|re^{i\theta}-e^{it}|^2} < \frac{1}{2\pi}\frac{(1-r)\cdot 2}{\left(\frac{\delta}{2}\right)^2} = \frac{1-r}{\pi\cdot\frac{\delta^2}{4}}$$

$\to 0$ unif as $r \nearrow 1$

Picture:



4) $\Delta_{(r,\theta)} P(r,\theta,t) \equiv 0 \qquad (0 \le r < 1).$

Pf: formula. Do it!

Big moment: Write $\quad u(r,\theta) = \int_0^{2\pi} P(r,\theta,t)\, f(t)\, dt$

u solves the Dirichlet problem!

Pf:
$$\left| u(r,\theta) - f(\theta_0) \right| = \left| u(r,\theta) - \underbrace{\left(\int_0^{2\pi} P(r,\theta,t)\, dt \right)}_{=1} f(\theta_0) \right|$$

$$= \left| \int_0^{2\pi} P(r,\theta,t) \left[f(t) - f(\theta_0) \right] dt \right|$$

Let $\varepsilon > 0$.
f cont. So $\exists\, \delta > 0$
such that $|f(t) - f(\theta_0)|$
$< \frac{\varepsilon}{2}$ when $|t - \theta_0| < \delta$

$$\le \int_{I_\delta} \underbrace{P(r,\theta,t)}_{>0} \left| \left[f(t) - f(\theta_0) \right] \right| dt$$

$$+ \int_{\theta_0 - \delta}^{\theta_0 + \delta} P(r,\theta,t) \left| \left[f(t) - f(\theta_0) \right] \right| dt$$

$$\leq \frac{(1-r)}{\pi \cdot \frac{\delta^2}{4}} \cdot \underset{\uparrow\ \max |f|}{2M} \cdot \underbrace{(2\pi)}_{> \text{Length } I_\delta}$$

$$+ \underbrace{\int_{\theta_0-\delta}^{\theta_0+\delta} \underbrace{P(r,\theta,t)}_{>0} \frac{\varepsilon}{2}\, dt}_{< \frac{\varepsilon}{2} \underbrace{\int_0^{2\pi} P(r,\theta,t)\, dt}_{=1}}$$

Can make first half $< \frac{\varepsilon}{2}$ as $r \nearrow 1$ with $re^{i\theta} \in D_{\delta/2}(e^{i\theta_0})$. Done!