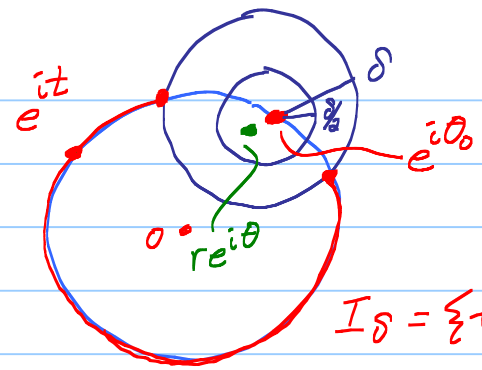
Lesson 14 Harmonic functions HWK 3 due Mon. 11:00pm in GS



$I_\delta = \{t : e^{it} \text{ in red}\}$

1) $$\left| u(r,\theta) - f(\theta_0) \right| = \left| \int_0^{2\pi} P(r,\theta,t)\left[f(t) - f(\theta_0)\right]dt \right|$$

$$= \left| \int_{I_\delta} \underbrace{P(r,\theta,t)}_{\text{small!}} \left[f(t) - f(\theta_0)\right]dt + \int_{\theta_0-\delta}^{\theta_0+\delta} P \underbrace{\left[f - f(\theta_0)\right]}_{\text{small!}} dt \right|$$

Conclude: $u(r,\theta)$ is continuous up to the boundary.

To make it continuous on $\overline{D_1(0)}$, define $u(1,\theta) = f(\theta)$.

Finally: $$\Delta_{(r,\theta)} u(r,\theta) = \Delta \int = \int_0^{2\pi} \underbrace{\Delta_{(r,\theta)} P(r,\theta,t)}_{\equiv 0 \quad r<1} f(t)\,dt$$

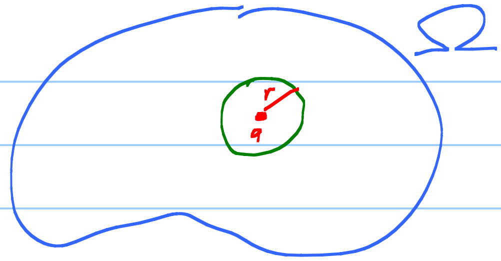
So u is harmonic on $D_1(0)$.

This proof is "Schwarz's theorem"

Thing to notice $$P(0,\theta,t) = \frac{1}{2\pi}\frac{1-0^2}{|0-e^{it}|} = \frac{1}{2\pi}$$

$$\underbrace{u(0)}_{\text{temp at } 0} = \int_0^{2\pi} \frac{1}{2\pi} \cdot f(\theta)\,d\theta = \text{ave of } f \text{ on unit circle}$$

Fact: Harmonic fcns satisfy the "averaging property"



$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta})\,d\theta$$

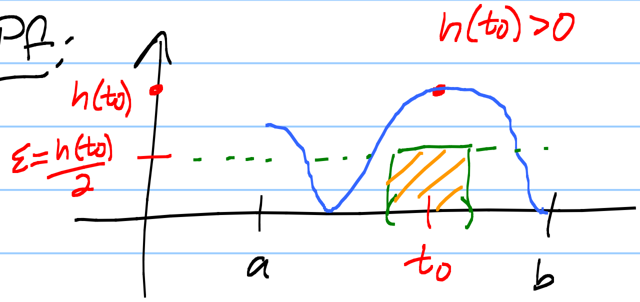
Why: Green's identities can be used to show this (later).

See link to "Something about Poisson and Dirichlet" on MA 428 home page.

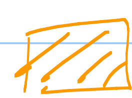
Fact: Ave property $\Longrightarrow$ Maximum principle

Calculus lemma: $0=\int_0^{2\pi}\sin t\,dt$ does not imply that $\sin t \equiv 0$, but if $h(t)$ is continuous and $h(t)\ge 0$ and $\int_a^b h(t)\,dt = 0$, then $h(t)\equiv 0$.

Pf:



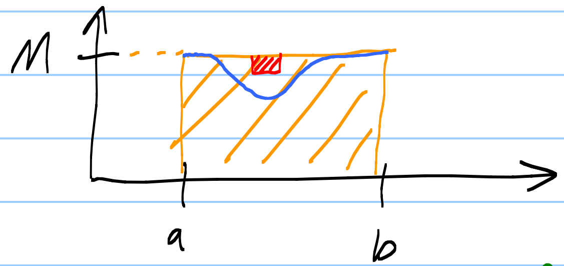
$\exists\, \delta>0$ such that $|h(t)-h(t_0)|<\varepsilon$ if $|t-t_0|<\delta$.

Aha! $\int_a^b h(t)\,dt >$ area  $\Rightarrow\!\Leftarrow$ ← contradiction

No such t_0. $h(t)\equiv 0$.

Lemma: If h is cont on $[a,b]$ and $h(t)\le M$ there, then if $\int_a^b h(t)\,dt = M(b-a)$, then $h(t) \equiv M$.

Pf: Apply first lemma to $\tilde{h}(t) = M - h(t)$.



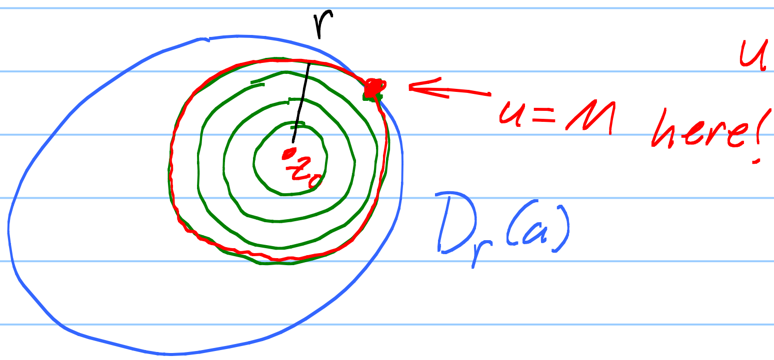
Max principle for harmonic fcns: If u is cont on $\overline{D_r(a)}$ and harmonic inside, then u assumes its max M at a pt on the boundary.

Pf: $\overline{D_r(a)}$ is a closed and bound set (compact).

Know a cont fcn on a closed bdd set assumes its max at a pt. $z_0 \in \overline{D_r(a)}$.

Case: z_0 in bndry. Done!

Case: $z_0 \in D_r(a)$ (inside).



$u(z_0) = M$, max on bndry.

Taking limits of expanding circles $\Longrightarrow$

$$\underbrace{u(z_0)}_{M} = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{u(z_0+re^{i\theta})}_{\le M}\, d\theta \le \underbrace{\frac{1}{2\pi}\cdot M \cdot 2\pi}_{M}$$

Calc lemma $\Rightarrow$ $u(z_0+re^{i\theta}) \equiv M$ on red circle.

<u>Big application</u>: Sol^n to D. problem is unique!

<u>Pf</u>: Two sol^{n}s u_1 and u_2. Then

$u_1 - u_2$ is a <u>harmonic</u> fcn, cont up to boundary and $\equiv 0$ on bndry.

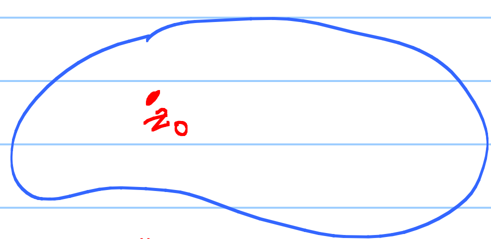
Max princ $\Rightarrow$ $\underset{\overline{D_1(0)}}{\text{Max}}\,(u_1-u_2) = \underset{C_1(0)}{\text{Max}}\,(u_1-u_2) = 0$

So $u_1 - u_2 \le 0$.

Repeat for $u_2 - u_1$. See $u_2 - u_1 \le 0$ too.

Conclude $u_1 \equiv u_2$.

<u>Second version of max princ</u>: If a harmonic fcn has a <u>local</u> max at a pt $z_0 \in \Omega$, then it must be <u>constant</u> on Ω.



Ω open, connected

z_0 = "hot spot"

Local max: $\exists\ D_\varepsilon(z_0) \subset \Omega$ such that $u(z) \le u(z_0)$ on $D_\varepsilon(z_0)$.

Another biggy:

Know $u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2 - r\cos(\theta - t)} f(t)\,dt$

Wow! Can differentiate under $\int$ infinitely many times in r and θ.

Consequently, harmonic fcns are C^∞-smooth!

Def^n: u is harmonic means that u is twice continuously diff'ble and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \equiv 0$.

Big fact: If a continuous fcn satisfies the ave prop on an open set, it is harmonic there.