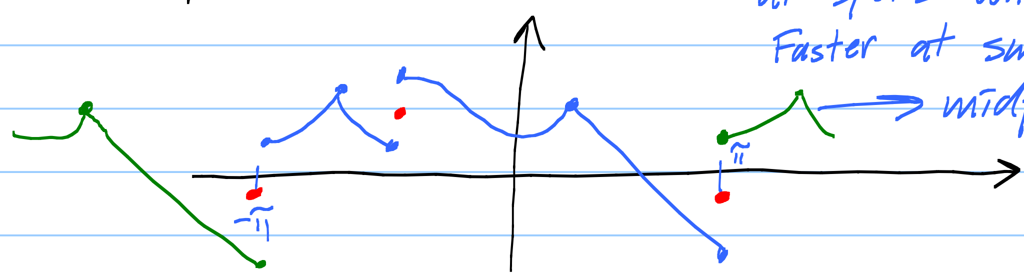
Lesson 17 The elephant in the room — the Riemann integral

What we know so far:

When f is piecewise $\mathcal{C}^1$-smooth:



Fourier series $\to f(x)$ at spots where f is continuous. Faster at smooth spots and $\to$ midpoint of jumps.

At $\pm\pi$, consider 2π-periodic extension for behavior.

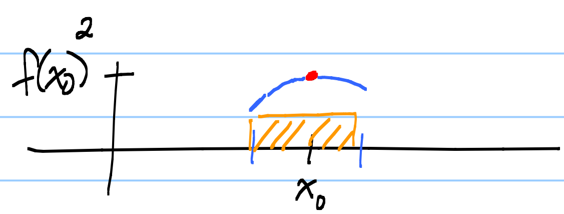
If f is merely piecewise continuous, then Parseval's identity holds.

(But we can't say that F.S. $\to f$ at any pt.!)

Cor: If f is piecewise continuous and all the Fourier coeff for f are zero, then $f(x_0)=0$ at each pt x_0 where f is continuous.

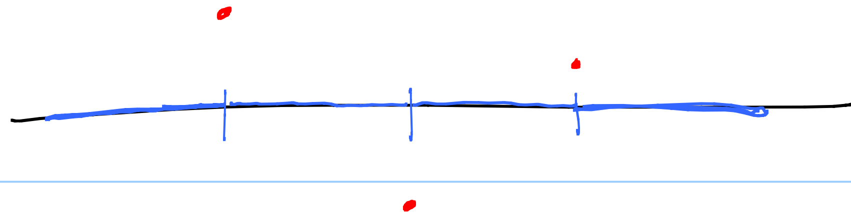
Philosophical point!

My proof: Parseval's: $\frac{1}{\pi}\int_{-\pi}^{\pi} |f|^2\,dx = 2a_0^2 + \underbrace{\sum_{n=1}^{\infty}(a_n^2+b_n^2)}_{=0}$



see $\int_{-\pi}^{\pi}|f|^2\,dx > (\text{area})$ ↯

See $f(x_0)$ must be zero at pts of continuity f:



Me: Bessel's, Fejér's, Parseval's, Cor

Stein: Fejér's, Cor, Bessel's, Parseval's

$L^1[a,b]$, $L^2[a,b]$:

Facts: 1) Cauchy-Schwarz inequality: $(f,g) = \int_a^b f(x)\overline{g(x)}\,dx$

$$|(f,g)| \le \|f\|\,\|g\|$$

where $\|f\| = \sqrt{\int_a^b |f(x)|^2\,dx}$ ← the L^2 norm of f

2) L^2 norm satisfies the triangle inequality

$$\|f+g\| \le \|f\| + \|g\|$$

Important for us:

$$\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \Big|f - \sum_{-N}^{N} \hat{f}(n)e^{inx}\Big|^2 dx}_{= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2\,dx - \sum_{-N}^{N} |\hat{f}(n)|^2} \le \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \Big|f - \sum_{-N}^{N} A_n e^{inx}\Big|^2 dx}_{f \text{ cont.}}$$

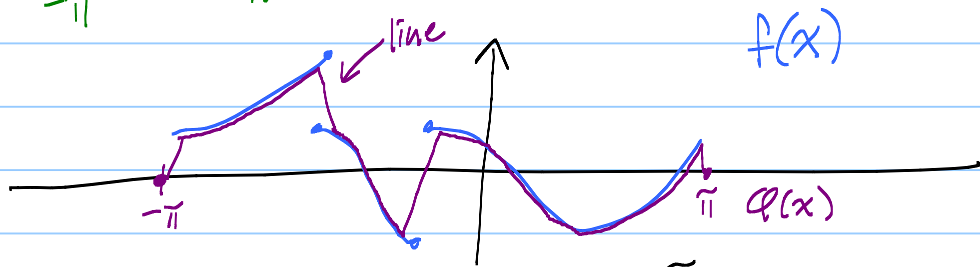
Can approx f unif by a trig poly.

Can make RHS as small as desired by Fejér's.

Get Parseval's for cont. fcns.

To get Parseval's for piecewise cont fcns.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f-\sum_{-N}^{N}\hat{f}(n)e^{inx}\right|^2 dx$$



2π-periodic extension!

Find cont fcn φ so that $\underbrace{\int_{-\pi}^{\pi}|f-\varphi|^2\,dx}_{\text{can make small!}} < \frac{\varepsilon}{2}$

$$\|\underbrace{f-\sum_{-N}^{N}\hat{f}(n)e^{inx}}_{\text{F.S.}}\| \le \left\|f-\sum_{-N}^{N}A_n e^{inx}\right\| = \left\|f-\varphi+\varphi-\sum_{-N}^{N}A_n e^{inx}\right\|$$

$$\le \underbrace{\|f-\varphi\|}_{<\frac{\varepsilon}{2}} + \underbrace{\|\varphi-F_N\|}_{\text{can make small}}$$

by taking trig poly F_N = Fejér poly for φ

Consequence: F.S. $\to f$ in $L^2[-\pi,\pi]$.

$$\left\|f-\sum_{-N}^{N}\hat{f}(n)e^{inx}\right\| \to 0 \quad \text{as } N\to\infty.$$

Ugly fact: f_n Riemann integrable and

f_n is a Cauchy seq in L^2, meaning

that, given $\varepsilon > 0$, there is an N such that

$$\|f_n - f_m\| < \varepsilon \quad \text{when} \quad n, m > N.$$

True that $f_n \xrightarrow{L^2} f_0 \in L^2$ in sense of Lebesgue.

Can happen that f_0 is not Riemann integrable!

Advanced fact: A fcn on $[a,b]$ is Riemann integrable if it is bounded and continuous "almost everywhere."

Def: "Almost everwhere" means at all $x \in [a,b]$ except on a set S of "measure zero".

Def^n S is a set of measure zero means that $\varepsilon > 0$, can find open intervals (α_n, β_n) such that $S \subset \bigcup_1^\infty (\alpha_n, \beta_n)$ and $\sum_1^\infty (\beta_n - \alpha_n) < \varepsilon$.

Cantor set

Keep taking out middle thirds.

What's left is a set of measure zero! It is uncountable like $\mathbb{R}$. It is perfect: every pt is a limit pt of the set.