Lesson 18 The hot wire problem

HWK 4 due Wed. 11pm in GS,
Exam 1 in class next Wed, March 1

The Riemann-Lebesgue Theorem: Fourier coeff $\to 0$ as $n \to \infty$

$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx \to 0 \quad \text{as } n \to \infty.$$

$$\int_{-\pi}^{\pi} f(x) \sin nx \, dx \to 0 \quad \text{as } n \to \infty.$$

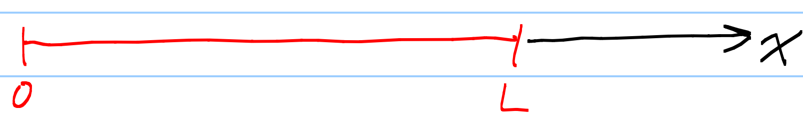
History: f is $\mathcal{C}'$-smooth, 2π-periodic, thm is very easy. (Int by parts: $dv = \cos nx \, dx$, $v = \frac{1}{n} \sin nx$)

Next: Bessel's shows $a_n, b_n \to 0$ as $n \to \infty$ if f is piecewise continuous. (Fejér's too)

Riemann: If f is Riemann integrable $\left(\int_{-\pi}^{\pi} |f| \, dx < \infty\right)$ then $a_n, b_n \to 0$.

Lebesgue: If f is measurable and Lebesgue integrable $\left(\int_{-\pi}^{\pi} |f| \, dx < \infty\right)$, then $a_n, b_n \to 0$.
↑ $\mathcal{L}$ integral

Hot wire problem



$u(x,t)$ = temp at x at time t.

Heat eqn: $\dfrac{\partial u}{\partial t} = c\dfrac{\partial^2 u}{\partial x^2}$

Given initial temp: $u(x,0) = f(x)$ ← known

Boundary conditions: $\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$ Homog bndry cond

Sol^n: Try $u(x,t) = X(x)T(t)$

$$\frac{\partial u}{\partial t} = XT' \overset{\text{want}}{=} cX''T = \frac{\partial^2 u}{\partial x^2}$$

$$\frac{1}{c}\frac{T'}{T} = \frac{X''}{X} = \lambda, \text{ both } = \text{const } \lambda$$

T problem: $T' - \lambda c T = 0$

$$T' = \lambda c T$$

Sol^n $T = Ke^{\lambda c t}$ easy.

X prob: Boundary cond $u(0,t) = X(0)T(t) \overset{\text{want}}{\equiv} 0$

$u(L,t) = X(L)T(t) \equiv 0$

Hmmm. $T(t) \equiv 0$ yields zero $u \equiv 0$. Need

$$\begin{cases} X(0) = 0 \\ X(L) = 0 \end{cases}$$

Case $\lambda > 0$ Write $\lambda = k^2$.

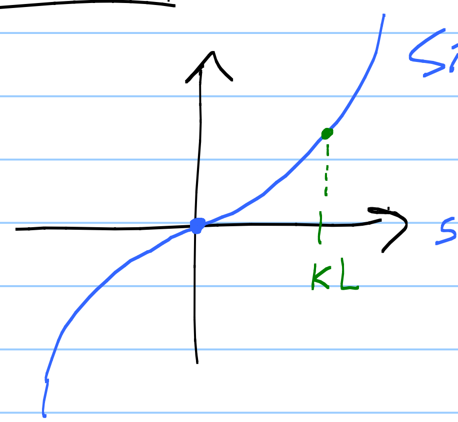
$$X'' - \lambda X = 0$$
$$r^2 - k^2 = 0$$
$$r = \pm k$$

$$X(x) = c_1 e^{kx} + c_2 e^{-kx}$$

$$X(0) = c_1 + c_2 \overset{\text{want}}{=} 0 \qquad \boxed{c_2 = -c_1}$$

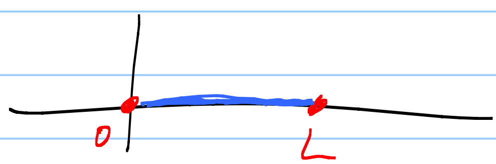
So $$X(x) = c_1 \underbrace{(e^{kx} - e^{-kx})}_{2\sinh kx}$$

$$X(L) = c_1 \underbrace{2\sinh kL}_{\neq 0.} \overset{\text{want}}{=} 0$$



Ouch. Get $c_1 = c_2 = 0$. Get $X(x) \equiv 0$.

Case $\lambda = 0$: $X'' = 0$. $X = Ax + B$ ← Line



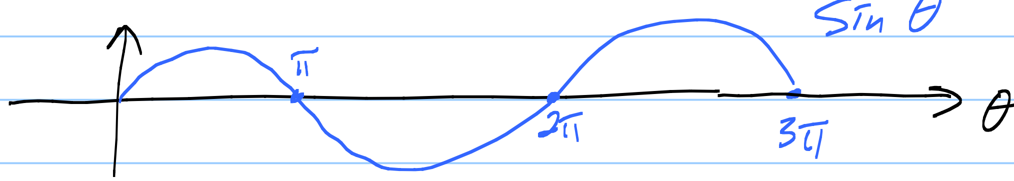
See $X(x) \equiv 0$.

Case $\lambda < 0$: $\lambda = -k^2$ $\qquad r^2 + k^2 = 0$

$$X(x) = c_1 \cos kx + c_2 \sin kx$$

$$X(0) = c_1 \overset{\text{want}}{=} 0 \qquad \text{So } c_1 = 0.$$

$$X(L) = c_2 \sin kL = 0$$



Need $KL = n\pi$, $n = 1, 2, 3, \ldots$

$$\boxed{K = \frac{n\pi}{L}} \qquad \lambda = -K^2 = -\left(\frac{n\pi}{L}\right)^2$$

Get $X(x) = c_2 \sin\frac{n\pi x}{L}$

$$T(t) = K e^{-c\left(\frac{n\pi}{L}\right)^2 t}$$

Get sol^n $u_n(x,t) = e^{-c\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi}{L}x$

Heat eqn is linear, boundary cond are homog (0), so superposition principle holds.

Sol^n $$u(x,t) = \sum_{n=1}^{\infty} b_n \underbrace{e^{-c\left(\frac{n\pi}{L}\right)^2 t}}_{\to 0 \text{ rapidly in } n,\ \to 0 \text{ in } t \text{ too!}} \sin\frac{n\pi}{L}x$$

Last thing: Initial cond:

want

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x = f(x)$$

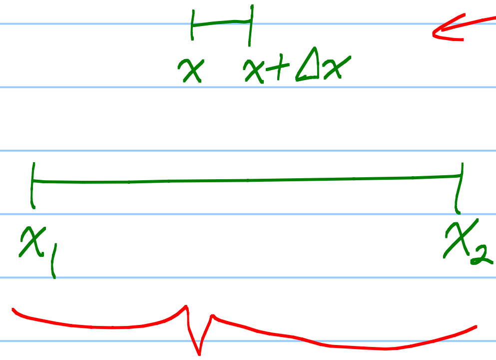
Invent F.S. on $[-L, L]$ or F. Sine series $[0, L]$.

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$$

Since $e^{-c\left(\frac{n\pi}{L}\right)^2 t} \to 0$ as $t \to \infty$, see

$u(x,t) \to 0$ as $t \to \infty$ too. Expected!

Next time: Derive heat eqn



← infinitesimal version

← contains "heat"

control "volume"