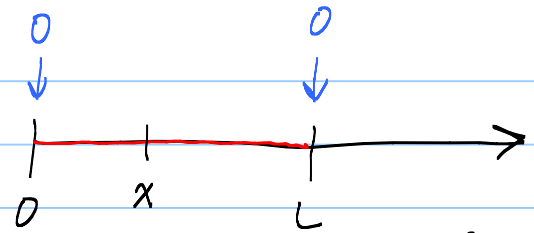
Lesson 19 The heat equation

HWK 4 due tonight 11 pm.
Find Exam 1 material
on home page

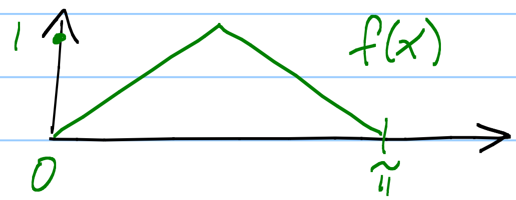


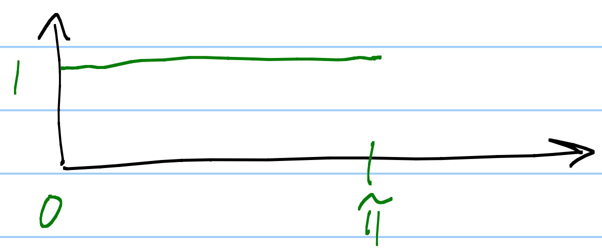
$$\frac{\partial u}{\partial t} = c\,\frac{\partial^2 u}{\partial x^2}$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n\, e^{-c\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi x}{L}$$

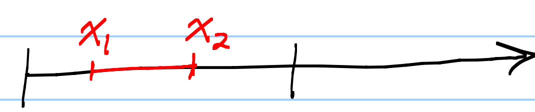
where $$B_n = \frac{2}{L}\int_0^L \underbrace{f(x)}_{\text{intial temp}} \sin\frac{n\pi x}{L}\,dx$$

Two problems:





How to derive the Heat eqn



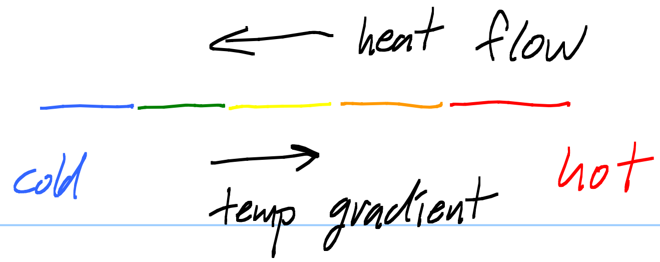
1 calorie is the amount of "heat" required to raise one gram of water 1 degree Celcius (= kelvin).

$$(\text{Heat in } (x_1,x_2)) = K\int_{x_1}^{x_2} u(x,t)\,dx$$

$$\text{Rate it changes} = \frac{\partial}{\partial t}\left[K\int_{x_1}^{x_2} u(x,t)\,dx\right]$$

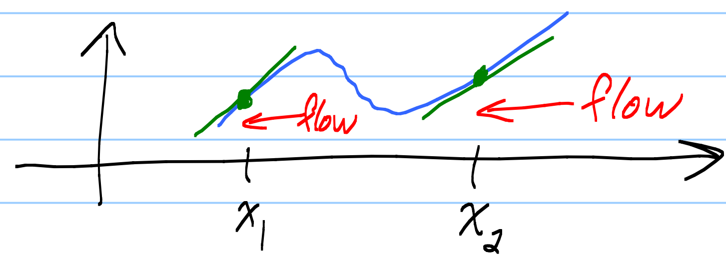
$$= K\int_{x_1}^{x_2} \frac{\partial u}{\partial t}(x,t)\,dx$$

Kelvin's law



$$(\text{Heat flow}) = -c\,(\text{temp gradient})$$

EX:



Rate of change:

$$\int_{x_1}^{x_2} k\,\frac{\partial u}{\partial t}(x,t)\,dx = \underbrace{c\,\frac{\partial u}{\partial x}(x_2,t) - c\,\frac{\partial u}{\partial x}(x_1,t)}_{\text{net heat inflow}}$$

$$= \int_{x_1}^{x_2} c\left[\frac{\partial^2 u}{\partial x^2}(x,t)\right]dx$$

So
$$\int_{x_1}^{x_2} \underbrace{\left[k\,\frac{\partial u}{\partial t} - c\,\frac{\partial^2 u}{\partial x^2}\right]}_{\text{must } \equiv 0} dx = 0$$

Aha! But x_1 and x_2 are arbitrary!

This shows integrand must be zero.

3 dim version

Rate of change of heat content

$$\frac{\partial}{\partial t}\left(k \iiint_V u(x,t)\, dV_x \right)$$

$$= \text{Net heat inflow} = c \underbrace{\iint_S \vec{\nabla u} \cdot \hat{n}\, d\sigma}_{\text{Flux}}$$

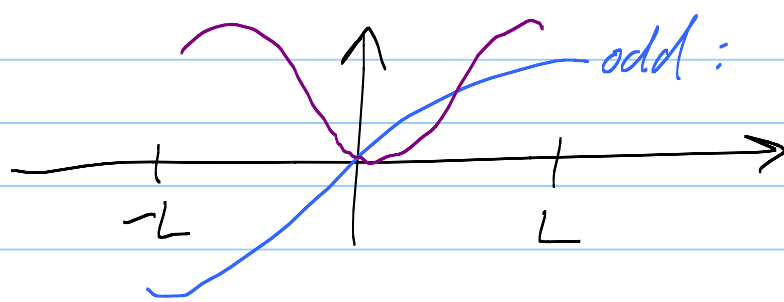
$$= c \iiint_V \underbrace{\mathrm{Div}(\vec{\nabla u})}_{\Delta u}\, dV$$

$$\iiint_V \Big[\underbrace{k \frac{\partial u}{\partial t} - c\,\Delta u}_{\text{must } \equiv 0} \Big]\, dV = 0 \quad \text{for } \underline{\text{any}} \text{ control volume.}$$

See Stein for Fourier series on $[-L, L]$

Orthogonal system: $1,\ \cos\frac{n\pi x}{L},\ \sin\frac{n\pi x}{L}$

even: no sine terms



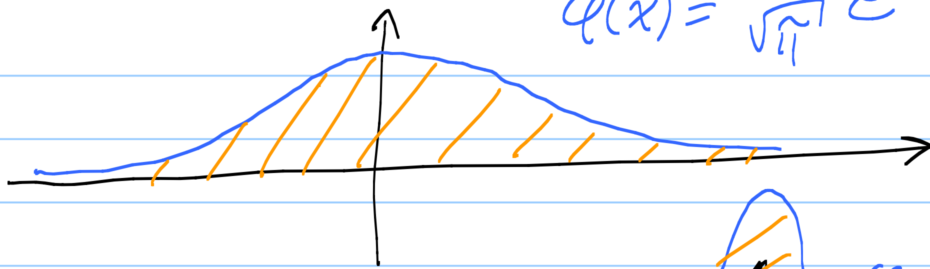
odd: no 1, cos terms

Get Fourier sine series and cosine series $[0, L]$

by taking odd (for sine) and even (for cosine) extensions.

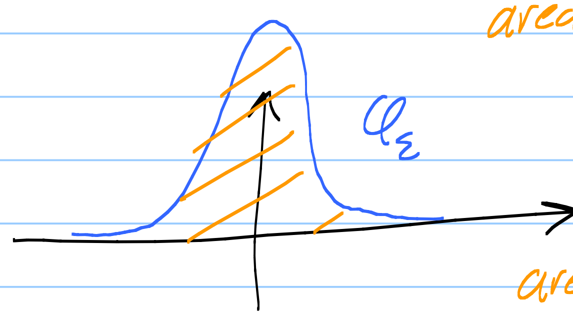
Friday: Good Kernels

$$\varphi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} \quad \text{(Gaussian)}$$



area = 1

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$$



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