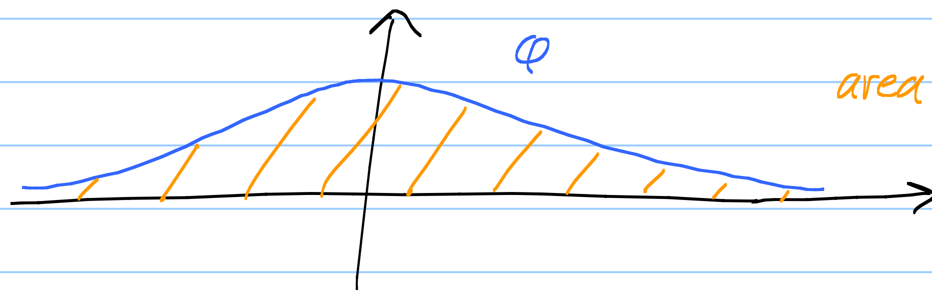
Lesson 20 Wonderful "good" kernels

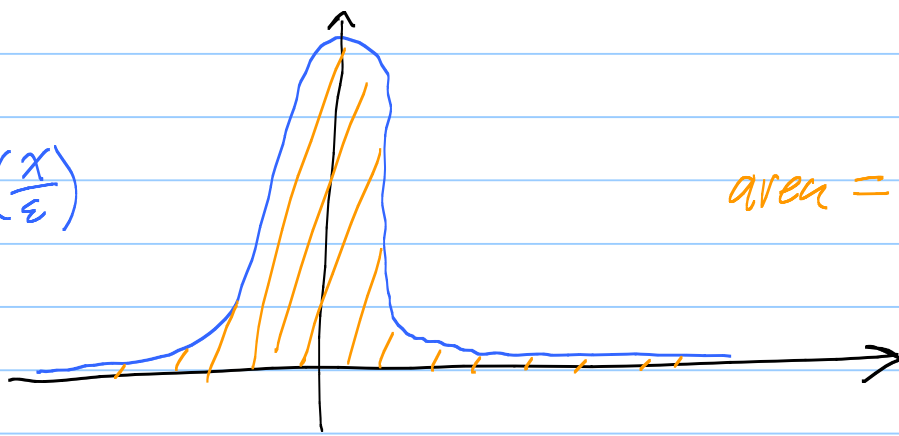
Exam 1 in class next Wed. See home page for info about it.

$\varphi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$



area = 1

$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$



area = 1

Famous calculation:

$$\left(\int_0^\infty e^{-x^2}\,dx\right)^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right)$$

$$= \underbrace{\int_0^\infty \int_0^\infty}_{\text{1st quadrant}} \underbrace{e^{-x^2} e^{-y^2}}_{e^{-(x^2+y^2)} = e^{-r^2}} \underbrace{dx\,dy}_{r\,dr\,d\theta}$$

$$= \int_0^{\pi/2} \int_0^\infty e^{-r^2}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\left[-e^{-u}\right]_0^\infty d\theta = \frac{\pi}{4}$$

$u = r^2$

So $\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 = \left(2\int_0^{\infty}\right)^2 = 4\cdot\frac{\pi}{4} = \pi$ ✓

Famous use of φ_ε: Convolution with a "bump" fcn.

We've seen $\displaystyle\int_{-\pi}^{\pi} \underbrace{F_N(t)\, f(x-t)}_{2\pi\text{-periodic}}\, dt$ ← "convolution"

Fejér kernel

On $\mathbb{R}$: $\displaystyle (\varphi_\varepsilon * f)(x) = \int_{-\infty}^{\infty} \varphi_\varepsilon(t)\, f(x-t)\, dt$

Fact: $\varphi_\varepsilon * f = f * \varphi_\varepsilon$

Feeling: $\varphi_\varepsilon * f$ smooth out f and approximates it as $\varepsilon \to 0$.

Why: $\displaystyle (\varphi_\varepsilon * f)(x) = \int_{-\infty}^{\infty} \varphi_\varepsilon(t)\, f(x-t)\, dt$

$\|$

$\displaystyle (f * \varphi_\varepsilon)(x) = \int_{-\infty}^{\infty} f(t)\, \varphi_\varepsilon(x-t)\, dt$

$$DQ = \frac{(f*\varphi_\varepsilon)(x+h) - (f*\varphi_\varepsilon)(x)}{h} = \int_{-\infty}^{\infty} f(t)\left[DQ \text{ on } \varphi_\varepsilon\,!\right] dt$$

$$\frac{d^n}{dx^n}(\varphi_\varepsilon * f) = f * \left(\frac{d^n}{dx^n}\varphi_\varepsilon\right)$$

If f is continuous and bounded (or even of "exponential type", meaning that $|f(x)| \le K e^{c|x|}$ on $\mathbb{R}$), see that $\varphi_\varepsilon * f$ is C^∞-smooth.

<u>Claim</u>: If f is continuous and bounded, then

$$\varphi_\varepsilon * f \to f \text{ \underline{uniformly} on } \mathbb{R}.$$

<u>Cor</u>: C^∞-smooth functions are "dense" among bounded continuous fcns on $\mathbb{R}$.

<u>Why</u>:

$$(\varphi_\varepsilon * f)(x) - f(x) \cdot \int_{-\infty}^{\infty} \varphi_\varepsilon(t)\, dt$$

$$= \int_{-\infty}^{\infty} \underset{\substack{\uparrow \\ \text{small} \\ \text{away from } t=0}}{\varphi_\varepsilon(t)} \underbrace{\big[f(x-t) - f(x) \big]}_{\text{small near } t=0} dt = I(x)$$

$$= \int_{-\delta}^{\delta} \varphi_\varepsilon(t) \underbrace{\big[f(x-t) - f(x)\big]}_{|\text{small}| < \frac{\varepsilon}{2}} dt + \left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \right) \ldots$$

So

$$|I(x)| \le \underbrace{\int_{-\delta}^{\delta} \varphi_\varepsilon(t) \cdot \frac{\varepsilon}{2}\, dt}_{< \frac{\varepsilon}{2} \underbrace{\int_{-\infty}^{\infty} \varphi_\varepsilon(t)\, dt}_{=1}} + \ldots$$

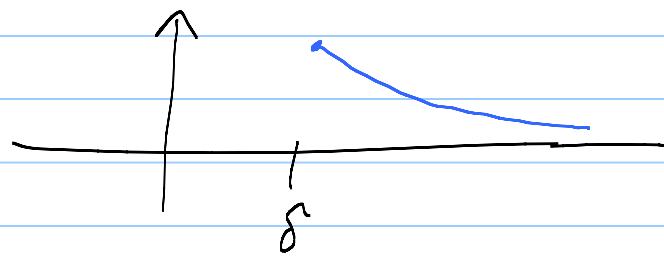
$$< \frac{\varepsilon}{2}$$

and $$\left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty}\right) \underset{\text{even}}{\varphi_\varepsilon(t)} \underbrace{|f(x-t)-f(x)|}_{< 2M \text{ where } |f|<M} \, dt$$

$$< (2M)\, 2 \underbrace{\int_\delta^\infty \varphi_\varepsilon(t)\, dt}_{\to 0 \text{ as } \varepsilon \to 0}$$

Can make second half $< \frac{\varepsilon}{2}$ by letting $\varepsilon \to 0$.

How to estimate $\displaystyle\int_\delta^\infty \frac{1}{\varepsilon} e^{-t^2/\varepsilon^2}\, dt$



$\varepsilon < \delta$

$e^{-t^2/\varepsilon^2} < e^{-t/\varepsilon}$

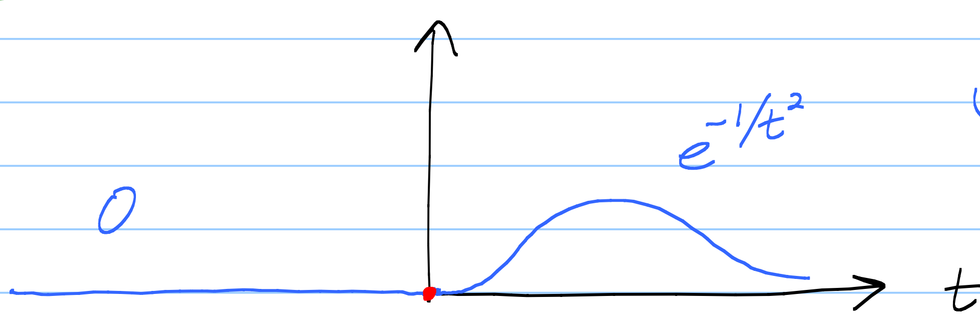
when $t > \delta$.

Can calculate

$\displaystyle\int_\delta^\infty e^{-t/\varepsilon}\, dt$

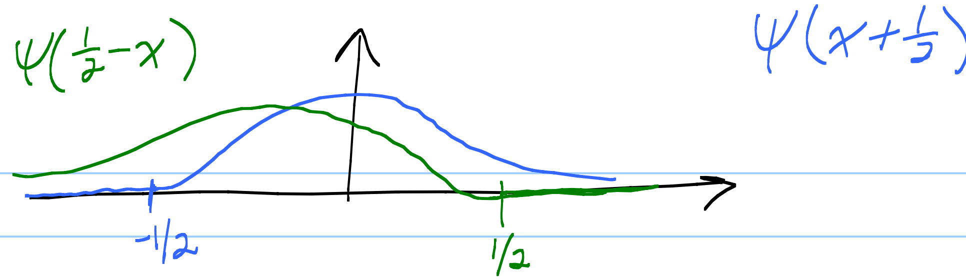
and see $\to 0$ fast!

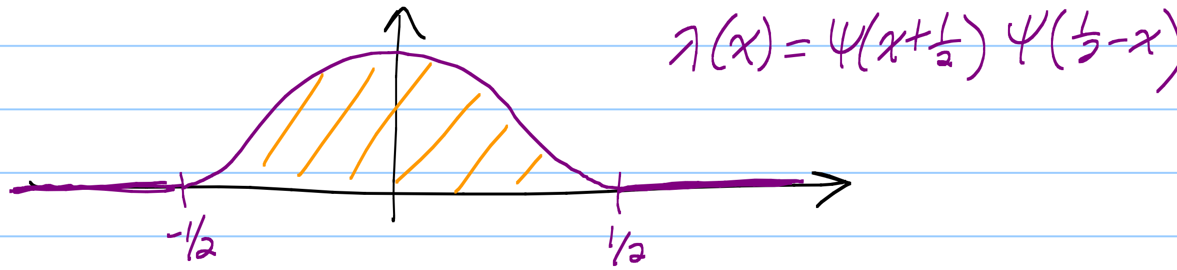
Better kernel   Great trick



ψ is C^∞-smooth

Trick



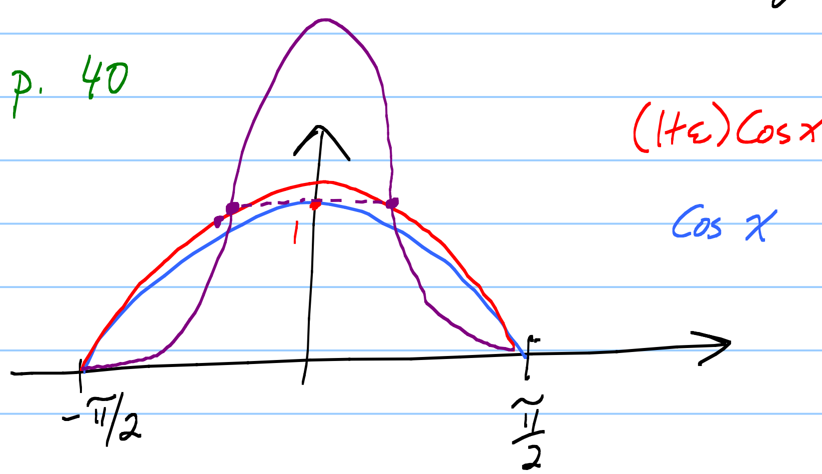


Let $c = \text{area} = \int_{-1/2}^{1/2} \lambda \, dx$,

Define $\varphi(t) = \frac{1}{c}\lambda(t)$ to get area $= 1$.

$\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi(t/\varepsilon)$ Fantastic good kernel.

Reading Stein p. 40



$$c\left[(1+\varepsilon)\cos x\right]^k = \text{bump fcn}$$

Fact: poly $P(\cos\theta, \sin\theta) = P\left(\frac{e^{i\theta}+e^{-i\theta}}{2}, \frac{e^{i\theta}-e^{-i\theta}}{2i}\right)$

= expand, simplify = trig poly!

<u>Stein</u>: If $\hat{f}(n) = 0$ for all n, then

$f \perp$ (trig polys)

$\Rightarrow f \perp$ bump fcn

$\Rightarrow f = 0$ at pts of continuity.