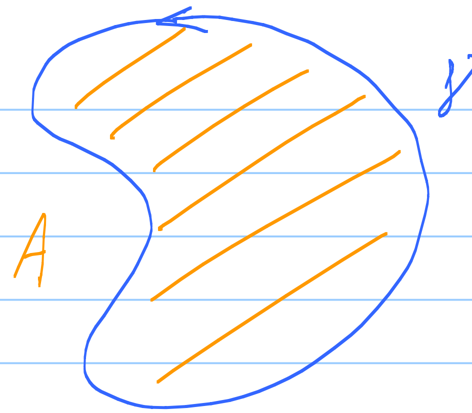
Lesson 22 The isoperimetric inequality via Fourier series!

γ simple closed curve, length L

Area $A \le \dfrac{L^2}{4\pi}$, = only when circle



Circle case $A = \pi r^2$

$L = 2\pi r \leftarrow r = \dfrac{L}{2\pi}$

$A = \pi \left(\dfrac{L}{2\pi}\right)^2 = \dfrac{L^2}{4\pi}$

Key ingredients

1) Green's theorem

$$\int_\gamma P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$Q = x \qquad P = -y$

$(\qquad) = 2$

$$\int_\gamma -y\,dx + x\,dy = \iint_\Omega 2\,dx\,dy = 2(\text{Area}(\Omega))$$

2) $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$F(x) = A_0 + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx)$

Parseval's $\Longrightarrow$ $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)F(x)\,dx = 2a_0A_0 + \sum_{n=1}^{\infty}(a_nA_n + b_nB_n)$

Pf: Parseval's for $\quad f+F$

$- \quad f-F$

New formula ✓

Complex version: $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\overline{F(x)}\,dx = \sum_{n=-\infty}^{\infty} c_n \overline{C_n}$

Steps to simplify calculations

1) Rescale so that $L = 2\pi$.
2) Parametrize γ : $z(t) = (x(t), y(t))$ via arc length t, $-\pi \le t \le \pi$
3) Assume γ is C^2-smooth.

Want to see $A \le \frac{L^2}{4\pi} \leftarrow \frac{(2\pi)^2}{4\pi} = \pi$

Notation: $x(t)$, $y(t)$

$$\hat{x}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t) e^{-int}\, dt, \quad \hat{y}(n)$$

We know $\hat{\dot{x}}(n) = in\,\hat{x}(n)$, $\hat{\dot{y}}(n) = in\,\hat{y}(n)$

Observation: Since t is arc length: $\frac{ds}{dt} = 1$

$$ds = \underbrace{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}_{=1}\, dt$$

So $\boxed{\dot{x}(t)^2 + \dot{y}(t)^2 \equiv 1}$

and $\int_{-\pi}^{\pi} \dot{x}(t)^2 + \dot{y}(t)^2\, dt = 2\pi$

$$\overset{\text{Parseval's}}{=} 2\pi \sum_{-\infty}^{\infty} \left(|in\,\hat{x}(n)|^2 + |in\,\hat{y}(n)|^2 \right)$$

Get $\boxed{\pi \cdot 1 = \pi \sum_{-\infty}^{\infty} n^2\left(|\hat{x}(n)|^2 + |\hat{y}(n)|^2\right)}$

Recall

$$A = \frac{1}{2}\int_\gamma -y\,dx + x\,dy$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} -y(t)\,\dot{x}(t) + x(t)\,\dot{y}(t)\,dt$$

$$= \frac{1}{2}\cdot 2\pi \sum_{-\infty}^{\infty}\left(-\hat{y}(n)\,\overline{(in\hat{x}(n))} + \hat{x}(n)\,\overline{(in\,\hat{y}(n))}\right)$$

Want to see $A \le \pi$.

$$\pi - A = \pi\sum_{-\infty}^{\infty} n^2\left(|\hat{x}(n)|^2 + |\hat{y}(n)|^2\right)$$
$$- \pi \sum_{-\infty}^{\infty} in\left(\hat{y}(n)\overline{\hat{x}(n)} - \hat{x}(n)\overline{\hat{y}(n)}\right)$$

Mind blowing thing: This expression is a sum of squares!

Write $\hat{x}(n) = \alpha_n + i\beta_n$

$\hat{y}(n) = a_n + i\,b_n$

Algebra $\pi - A = \pi \sum_{-\infty}^{\infty}\left[n^2\left(\alpha_n^2 + \beta_n^2 + a_n^2 + b_n^2\right) - 2n\left(\beta_n a_n - \alpha_n b_n\right)\right]$

$$= \pi \sum_{-\infty}^{\infty} \left[(n\alpha_n + b_n)^2 + (n\beta_n - a_n)^2 + \underbrace{(n^2-1)}_{\substack{=0 \\ n=\pm 1}} (a_n^2 + b_n^2) \right]$$

≥ 0 because sum of squares!

What if $=$? Each term in sum must $= 0$.

See $a_n = 0$, $b_n = 0$ when $|n| > 1$ from last term.

Now see $\alpha_n = 0$, $\beta_n = 0$ when $|n| > 1$ too from first two squares.

Aha!

$$\left. \begin{aligned} x(t) &= c_0 + c_{-1} e^{-it} + c_1 e^{it} \\ y(t) &= C_0 + C_{-1} e^{-it} + C_1 e^{it} \end{aligned} \right\}$$

Show this is a circle!

Not hard.

Do it. $|n| > 1$ terms. Gone.

Traslate curve to make $\hat{x}(0)$, $\hat{y}(0)$ both $= 0$.

$n=1$ term $\quad \underbrace{(1\cdot\alpha_1 + b_1)^2}_{\alpha_1 = -b_1} + \underbrace{(1\beta_1 - a_1)^2}_{\beta_1 = a_1} + \underbrace{(1^2-1)(a_1^2 + b_1^2)}_{=0}$

$n=-1$ term $\quad \underbrace{(-1\,\alpha_{-1} + b_{-1})^2}_{b_{-1} = \alpha_{-1}} + \underbrace{(-1\,\beta_{-1} - a_{-1})^2}_{a_{-1} = -\beta_{-1}} + \underbrace{(1^2-1)(a_{-1}^2 + b_{-1}^2)}_{=0}$

Bottom line:

$$\begin{cases} x(t) = 2\alpha_1 \cos t - 2\beta_1 \sin t \\ y(t) = 2\beta_1 \cos t + 2\alpha_1 \sin t \end{cases}$$

Circle!

Note: Because $x(t)$ and $y(t)$ are real,

$$\hat{x}(-n) = \overline{\hat{x}(n)}$$

$$\hat{y}(-n) = \overline{\hat{y}(n)}$$

Why: $$\hat{x}(-n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)\, e^{-i(-n)t}\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)\, \underbrace{e^{int}}_{=\overline{e^{-int}}}\, dt$$

$$= \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)\, e^{-int}\, dt}$$