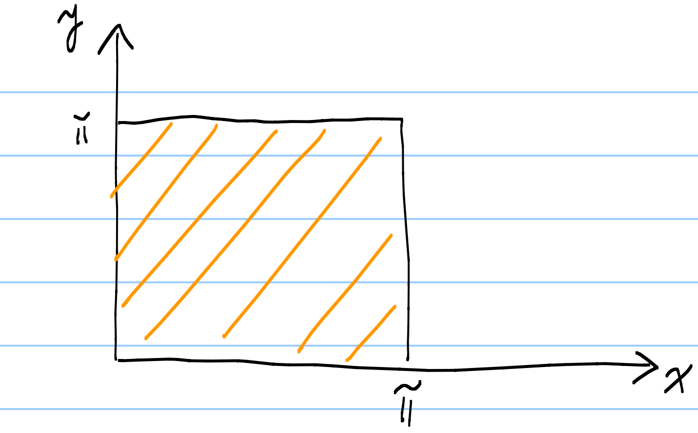
Lesson 25 Vibrating square drum, Laplace transform

HWK 5 due Wed



$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u \qquad (c=1)$$

$u(x,y,t)=0$ on frame

$$\begin{cases} u(x,y,0) = f(x,y) \\ \frac{\partial u}{\partial t}(x,y,0) = g(x,y) \end{cases}$$

$$u(x,y,t) = \sum_{n,m=1}^{\infty} \sin nx \sin my \left(A_{nm} \cos w_{nm} t + B_{nm} \sin w_{nm} t \right)$$

where $w_{nm} = \sqrt{n^2+m^2}$

Need $$\begin{cases} u(x,y,0) = \sum_{n,m=1}^{\infty} A_{nm} \sin nx \sin my \leftarrow \text{Get } A\text{'s} \\ \frac{\partial u}{\partial t}(x,y,0) = \sum_{n,m=1}^{\infty} w_{nm} B_{nm} \sin nx \sin my \leftarrow \text{Get } B\text{'s} \end{cases}$$

Need $$\sum_{n=1}^{\infty} \underbrace{\left(\sum_{m=1}^{\infty} A_{nm} \sin my \right)}_{\text{Fourier sine series coeff for } f(\cdot,y)} \sin nx \overset{\text{want}}{=} f(x,y)$$

See $$\left(\sum_{m=1}^{\infty} A_{nm} \sin my \right) = \underbrace{\frac{2}{\pi} \int_0^{\pi} f(x,y) \sin nx \, dx}_{= F(y)}$$

$$A_{nm} = \frac{2}{\pi} \int_0^{\pi} F(y) \sin my \, dy$$

$$A_{nm} = \frac{2}{\pi} \int_0^{\pi} \left(\frac{2}{\pi} \int_0^{\pi} f(x,y) \sin nx \, dx \right) \sin my \, dy \overset{\text{Fubinni's}}{=}$$

$$\boxed{A_{nm} = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} f(x,y) \sin nx \sin my \, dx \, dy}$$

Similarly, $B_{nm} = \dfrac{4}{\pi^2 \omega_{nm}} \displaystyle\int_0^{\pi} \int_0^{\pi} g(x,y) \sin nx \sin my \, dx \, dy$

Alternatively: $\underbrace{\sin nx \sin my}_{\varphi_{nm}}$ are orthogonal:

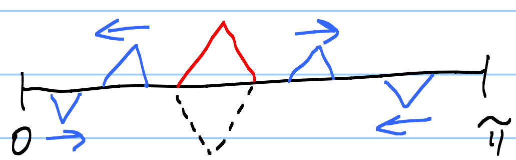
$$\langle \varphi_{nm}, \varphi_{k\ell} \rangle = \int_0^{\pi} \int_0^{\pi} \varphi_{nm}(x,y) \varphi_{k\ell}(x,y) \, dx \, dy$$

$$= \begin{cases} 0 & (n,m) \neq (k,\ell) \\ \neq 0 & (n,m) = (k,\ell) \end{cases}$$

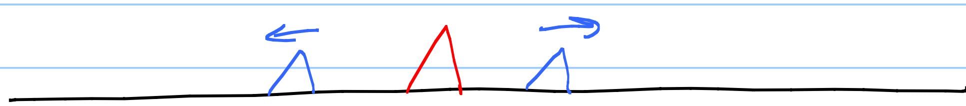
Fun facts: $\omega_{nm} = \sqrt{n^2+m^2}$ discordant overtones!

Another strange thing: String: $\omega_n = n \leftarrow \cos nt$ 2π-periodic

Sq. drum: ω_{nm} don't have a common period



Initial shape gets repeated!



infinite string

Huygen's principle: waves in $\mathbb{R}^1, \mathbb{R}^3, \mathbb{R}^5, \dots$ go away, leave space in peace

$\mathbb{R}^2, \mathbb{R}^4, \mathbb{R}^6, \dots$ they don't

Laplace transform: $f(x) \longmapsto F(s) = \int_0^\infty f(x)\, e^{-sx}\, dx$

Useful for studying initial value problems at $x=0$ for ODEs.

Turns linear ode's with const coeff into algebra!

Big facts: $\mathcal{L}[f(x)] = F(s)$ is a linear operator.

1) $\mathcal{L}[f'] = sF(s) - f(0)$

2) $\mathcal{L}[f''] = s^2 F(s) - s f(0) - f'(0)$

3) Can undo the Laplace transform!

If $\mathcal{L}[f_1] = \mathcal{L}[f_2]$ for all $s > M$ (for some M),

then $f_1 \equiv f_2$ on $[0, \infty)$.

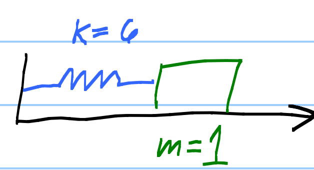
Consequently, $\mathcal{L}^{-1}$ makes sense.

Plan: $\mathcal{L}[\text{ODE}]$ = Algebra in $F(s)$

Solve for F. Sol^n: $f(x) = \mathcal{L}^{-1}[F(s)]$.

EX: $y'' + 5y' + 6y = e^{-4t}$

c=5 friction term



$m y'' = -6y - 5y' + e^{-4t}$

Hooke's friction external

$\mathcal{L}[y'' + 5y' + 6y] = \mathcal{L}[e^{-4t}]$

$$\left(s^2 Y(s) - s y(0) - y'(0)\right) + 5\left(s Y(s) - y(0)\right) + 6Y = \frac{1}{s+4}$$

$$\underbrace{(s^2 + 5s + 6)}_{(s+2)(s+3)} Y(s) = s y(0) + y'(0) + 5 y(0) + \frac{1}{s+4}$$

Plug in initial conditions. Solve for Y.

Take $y(0) = 0$, $y'(0) = 0$

Get $Y(s) = \dfrac{1}{(s+2)(s+3)(s+4)}$

$$\overset{\text{P.F.}}{=} \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+4}$$

$$y(t) = Ae^{-2t} + Be^{-3t} + Ce^{-4t}$$