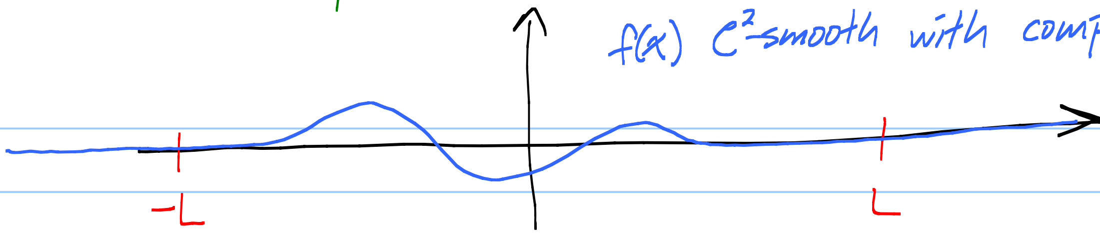
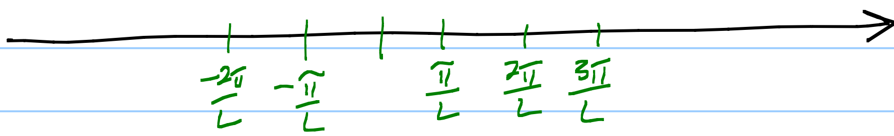
Lesson 28 The complex Fourier transform



$f(x)$ C^2-smooth with compact support $\subset (-L, L)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \qquad \text{where} \qquad c_n = \frac{1}{2L}\int_{-L}^{L} f(t)\, e^{-in\pi t/L}\, dt$$

$$= \frac{1}{\pi}\sum_{n=-\infty}^{\infty} \Bigg(\frac{1}{2}\cdot \underbrace{\frac{\pi}{L}}_{\Delta w} \int_{-\infty}^{\infty} f(t)\, e^{-i \overbrace{\frac{n\pi}{L}}^{w_n} t}\, dt \Bigg)\, e^{i \overbrace{\frac{n\pi}{L}}^{w_n} x}$$



$$\xrightarrow[L\to\infty]{} \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt \right) e^{iwx}\, dw$$

Let $\displaystyle F(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt$ $\leftarrow$ Complex Fourier transform

Write $F(w) = \tilde{\mathcal{F}}[f] = \hat{f}(w)$

Fourier inversion formula:

$$f(x) = \int_{-\infty}^{\infty} F(w)\, e^{iwx}\, dw$$

Write $f(x) = \tilde{\mathcal{F}}^{-1}[F] = \check{F}$

<u>Remark</u>: $\frac{1}{2\pi}$ can go into $\tilde{\mathcal{F}}$ or $\tilde{\mathcal{F}}^{-1}$ (but not both).

<u>Or</u> $\frac{1}{\sqrt{2\pi}}$ and $\frac{1}{\sqrt{2\pi}}$ into <u>both</u> $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^{-1}$.

Today: Do $\frac{1}{\sqrt{2\pi}}$ version.

Big properties

$$\tilde{\mathcal{F}}[f](s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx = \hat{f}(s)$$

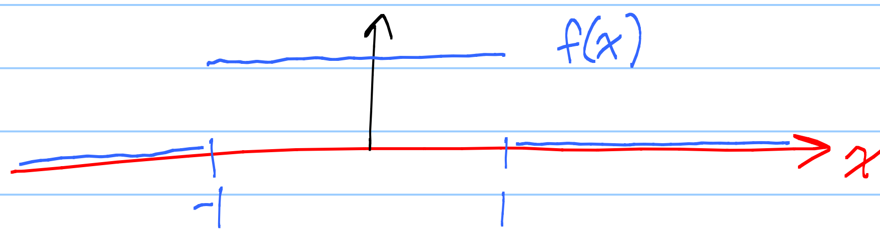
1) $\tilde{\mathcal{F}}[f'] = is\,\hat{f}(s)$

2) $\tilde{\mathcal{F}}[f''] = (is)(is)\hat{f}(s) = -s^2\hat{f}(s)$

3) If f is piecewise C^1-smooth and $\int_{-\infty}^{\infty} |f|\,dx < \infty$

$$f(x) \text{ "=" } \tilde{\mathcal{F}}^{-1}[\hat{f}] = \begin{cases} f(x) & f \text{ cont at } x \\ \text{midpoint of jump} & \text{at jumps} \end{cases}$$

Hot wire problem:



$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (c=1)$$

Initial cond: $u(x,0) = f(x)$

No boundary cond!

Big idea: Hit $u(x,t)$ with $\tilde{\mathcal{F}}$ in space var. x.

Write $\hat{u}(s,t) = \tilde{\mathcal{F}}[u] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)\, e^{-isx}\, dx$

Hmmm. Take $\frac{\partial}{\partial t}$ of this, and diff under $\int$.

$$\frac{\partial \hat{u}}{\partial t} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{\frac{\partial u}{\partial t}(x,t)}_{\frac{\partial^2 u}{\partial x^2}(x,t)}\, e^{-isx}\, dx$$

$$= \underbrace{\tilde{\mathcal{F}}\left[\frac{\partial^2 u}{\partial x^2}\right]}_{-s^2 \underbrace{\tilde{\mathcal{F}}[u]}_{\hat{u}}}$$

Get $\boxed{\dfrac{\partial \hat{u}}{\partial t} = -s^2 \hat{u}}$ ⟵ easy!

$$\hat{u} = \underbrace{C(s)}_{\text{const might depend on } s.} e^{-s^2 t}$$

Next: Get $C(s)$ from the initial cond:

$$\hat{u}(s,0) = C(s) \underbrace{e^{-s^2 \cdot 0}}_{=1} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{u(x,0)}_{f(x)} e^{-is \cdot x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1 \cdot e^{-isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-is} e^{-isx} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{-is} - e^{-is(-1)}}{-is} = \frac{2 \cdot 1}{\sqrt{2\pi}} \underbrace{\frac{e^{is} - e^{-is}}{2i}}_{\sin s} \cdot \frac{1}{s}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin s}{s}$$

So $\hat{u}(s,t) = \sqrt{\dfrac{2}{\pi}} \dfrac{\sin s}{s} e^{-s^2 t}$

Last step: Undo $\tilde{\mathcal{F}}$ using Fourier inversion formula

$$u(x,t) = \mathcal{F}^{-1}\left[\sqrt{\frac{2}{\pi}}\,\frac{\sin s}{s}\,e^{-s^2 t}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\left(\sqrt{\frac{2}{\pi}}\,\frac{\sin s}{s}\,e^{-s^2 t}\right)}_{F(s)} e^{isx}\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \underbrace{\frac{\sin s}{s}}_{\text{even}}\, \underbrace{e^{-s^2 t}}_{\text{even}} \left(\underbrace{\cos sx}_{\text{even}} + i \underbrace{\sin sx}_{\text{odd}}\right) ds$$

$$\int_{-\infty}^{\infty} = \lim_{L \to \infty} \int_{-L}^{L}$$

$$(\text{even})(\text{even})(\text{odd}) = \text{odd}$$

Imaginary part goes away!

Sol^n: $$\boxed{u(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s}\, e^{-s^2 t} \cos sx \, ds}$$