Lesson 29 The Heisenberg uncertainty principle

HWK 5 sol'ns posted on home page

Can't know both position and momentum of a particle.

Math version: f and $\hat{f}$ cannot both have compact support.

Why: Support of f = closure of the set of points x where $f(x) \neq 0$.

$$\operatorname{supp} f = \overline{\{x : f(x) \neq 0\}}$$

Compact means closed and bounded.

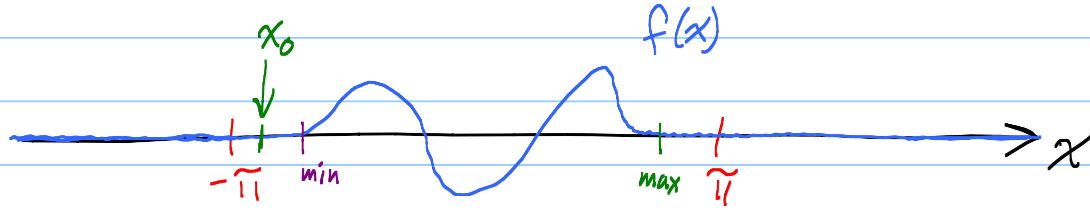
Closure of a set S = $S \cup \{$all limit pts of $S\}$

EX: $\overline{[0,1)} = [0,1]$. $\overline{\mathbb{Q}} = \mathbb{R}$. $\overline{\mathbb{Z}} = \mathbb{Z}$.

Topology: S closed $\Longleftrightarrow$ $\mathbb{R} - S$ is open

S compact means: every covering of S by open sets has a finite subcover. [If $S \subset \bigcup_{\alpha \in \Omega} \mathcal{O}_\alpha$, then $S \subset \bigcup_{n=1}^{N} \mathcal{O}_{\alpha_n}$]

Case 1: $\operatorname{supp} f \subset (-\pi, \pi)$, f C^1-smooth



$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)\, e^{-isx}\, dx$$

Aha! If $\hat{f}$ has compact support too then $\hat{f}(s) = 0$, $|s| > M$.

... and aha! Fourier coeff $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx = 0$

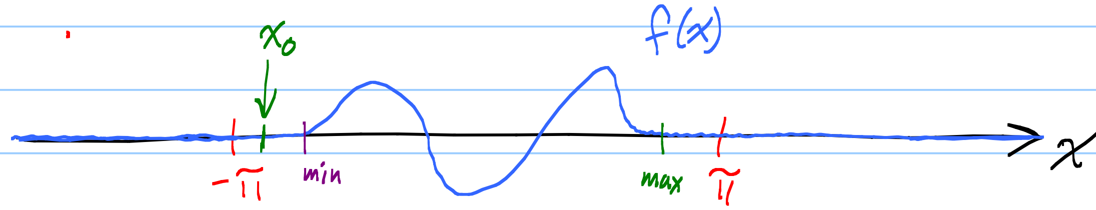
when $|n| > M$.

Aha! On $[-\pi,\pi]$, $\underbrace{f(x)}_{\text{real}} = \sum_{-\infty}^{\infty} c_n e^{inx} = \underbrace{\sum_{-N}^{N} c_n e^{inx}}$ Trig poly!

$(N \le M < N+1)$

$$= \sum_{-N}^{N} (a_n + i b_n)[\cos nx + i \sin nx]$$

$$= \underbrace{\left(\sum_{-N}^{N} \underset{\text{real}}{r_n} \cos nx + \underset{\text{real}}{R_n} \sin nx\right)}_{\text{real part of trig poly}} + i \underbrace{\left(\text{who cares}\right)}_{\text{must } \equiv 0}$$



Idea: Use power series
$$\begin{cases} \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \\ \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \end{cases}$$

R. of C. $= \infty$

$f(x) =$ Real trig poly on $[-\pi,\pi]$ = Power series $\sum_{n=0}^{\infty} A_n (x-x_0)^n$

R. of C. $= \infty$.

Aha! $A_n = \frac{f^{(n)}(x_0)}{n!} = 0$ for all n.

What the heck? I've shown that $f \equiv 0$ on $[-\pi,\pi]$.

[Know $f=0$ outside $[-\pi,\pi]$ too.] So $f \equiv 0$, and $\hat{f} \equiv 0$.

Remark: Can improve this argument to allow f to be merely continuous using Fejér's thm. Can still show $f =$ Trig poly on $[-\pi, \pi]$.

Hmmm. How to handle $\operatorname{supp} f \subset [-L, L]$?

Fourier series on $[-L, L]$:
$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i \frac{n\pi x}{L}} \, dx$$
$$\int_{-\infty}^{\infty}$$

$= 0 \qquad \left|\frac{n\pi}{L}\right| > M$!

Same argument works!

Or dilate variables.

Fact: Let $f_\delta(x) = f(\delta x)$.

$$\widehat{f_\delta}(s) = \frac{1}{\delta} \hat{f}\left(\frac{s}{\delta}\right)$$

$$\hat{f_\delta}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\underbrace{\delta x}_{u}) e^{-isx} \, dx$$

$u = \delta x \qquad x = \frac{u}{\delta}$

$du = \delta \, dx \qquad dx = \frac{1}{\delta} du$

$\begin{cases} -\infty < x < \infty \\ -\infty < u < \infty \end{cases}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-is\frac{u}{\delta}} \left(\frac{1}{\delta} du\right)$$

$$= \frac{1}{\delta} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-i \frac{s}{\delta} u} \, du \right)$$

$$= \frac{1}{\delta} \hat{f}\left(\frac{s}{\delta}\right)$$

Dilate using δ big so that $\frac{1}{\delta}[-L, L] \subset (-\pi, \pi)$. Use previous result to show $\frac{1}{\delta} \hat{f}(s/\delta) \equiv 0$. $\Longrightarrow \hat{f} \equiv 0$. ↯