Lesson 31 Integrable functions on $\mathbb{R}$

Change: HWK 6 due Tues at 11:00pm

EX: $$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \lim_{A\to-\infty}\int_A^0 \frac{\sin x}{x}\,dx + \lim_{B\to\infty}\int_0^B \frac{\sin x}{x}\,dx$$

but $$\int_{-\infty}^{\infty}\left|\frac{\sin x}{x}\right| dx = \infty \quad \text{(Conditionally convergent)}$$

Def^n f is absolutely integrable if $\int_{-\infty}^{\infty} |f|\,dx < \infty$.

Lemma: If f is absolutely integrable, then it is integrable:
$$\int_{-\infty}^{\infty} f\,dx = \lim_{A\to-\infty}\int_A^0 f(x)\,dx + \lim_{B\to\infty}\int_0^B f(x)\,dx$$

Important fact: If f is absolutely integrable, then $\int_B^{\infty} f\,dx$ and $\int_B^{\infty}|f|\,dx$ both $\to 0$ as $B\to\infty$.

Same for $\int_{-\infty}^{A}$ version as $A\to-\infty$.

Pf: Similar to $\left(\sum^{\infty} a_n \text{ exists}\right) \Longrightarrow a_n \to 0,\ n\to\infty$.

$$\int_a^{\infty}|f(x)|\,dx = \underbrace{\int_a^B |f(x)|\,dx}_{\xrightarrow[B\to\infty]{} \int_a^{\infty}|f|\,dx} + \underbrace{\int_B^{\infty}|f|\,dx}_{\text{must } \to 0 \text{ as } B\to\infty}$$

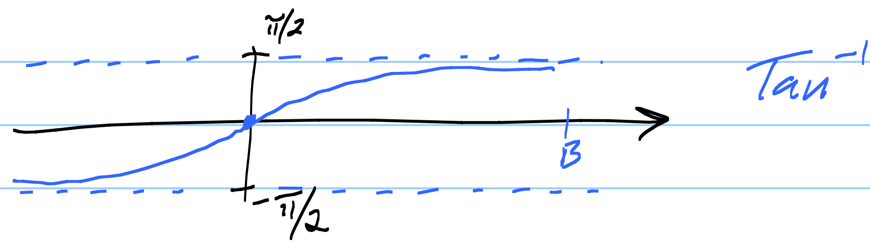
Note $$\left|\int_B^{\infty} f\,dx\right| \le \int_B^{\infty}|f|\,dx \to 0 \quad \text{as } B\to\infty.$$

Stein: $\mathcal{M}(\mathbb{R})$ = fcns of moderate decrease on $\mathbb{R}$.

[Note: We will always assume our fcns f are piecewise continuous.

$$\mathcal{M}(\mathbb{R}) = \{ f : f \text{ continuous and } \exists\, M>0,\ C>0 \text{ such that } |f(x)| \le \frac{C}{1+x^2} \text{ when } |x|>M \}$$

Note: $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{B\to\infty} \operatorname{Tan}^{-1} x \Big|_0^B = \frac{\pi}{2} - 0 = \frac{\pi}{2}$



Basic facts

1) $f \longmapsto \displaystyle\int_{-\infty}^{\infty} f\,dx$ is a linear operator. $f \in \mathcal{M}(\mathbb{R})$

2) $\displaystyle\int_{-\infty}^{\infty} f(x+h)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$

3) $\displaystyle\int_{-\infty}^{\infty} f(kx)\,dx = \frac{1}{k}\int_{-\infty}^{\infty} f(x)\,dx$

4) $\displaystyle\int_{-\infty}^{\infty} |f(x+h) - f(x)|\,dx \to 0$ as $h \to 0$.

Pf of 4: Given $\varepsilon > 0$, $\exists\, R > 0$ such that

$$\int_{|x|>R-1} |f|\,dx < \frac{\varepsilon}{4}, \qquad \text{Assume } |h| < 1.$$

Now $\displaystyle\int_{-\infty}^{\infty} |f(x+h)-f(x)|\,dx =$

$$\underbrace{\int_{-R}^{R} |f(x+h)-f(x)|\,dx}_{\text{Use uniform continuity of } f \text{ on } [-R,R].} + \underbrace{\int_{|x|>R} |f(x+h)-f(x)|\,dx}$$

Use <u>uniform</u> continuity of f on $[-R,R]$. Can make $\int_{-R}^{R} < \frac{\varepsilon}{4}$ if h is so small that $|f(x+h)-f(x)| < \frac{\varepsilon/4}{2R}$ on $[-R,R]$.

$$< \int_{|x|>R} |f(x+h)|\,dx + \int_{|x|>R} |f|\,dx$$

both $< \varepsilon/4$ by choice of R and fact that $h<1$.

<u>Cool thing about Fourier transform:</u>

$\mathfrak{F}[e^{-x^2}]$ is "basically" itself.

<u>Why</u>: $\displaystyle\mathfrak{F}[e^{-x^2}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2} e^{-isx}\,dx = F(s)$

Calculate $\displaystyle F'(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2} \frac{d}{ds}\left(e^{-isx}\right) dx$

safe! because $e^{-x^2} \to 0$ so rapidly at $\pm\infty$.

$$= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} -i\,2x\, e^{-x^2} e^{-isx}\,dx$$

etc., etc.

$$\ldots \quad F'(s) = -\tfrac{1}{2} s F(s)$$ Linear ODE!

$$\underset{\uparrow}{F'(s)} + \underbrace{\tfrac{1}{2}s}_{P(s)} F(s) = 0$$

Int. factor $u = e^{\int \frac{1}{2} s \, ds}$

$u = e^{s^2/4}$

1, standard form ✓

$$\underbrace{e^{s^2/4} \cdot \left[F' + \tfrac{1}{2} F \right]}_{= \frac{d}{ds}\left[e^{s^2/4} \cdot F \right]} = 0$$

So $\quad e^{s^2/4} \cdot F(s) \equiv C$, a const.

$$F(s) = C e^{-s^2/4}$$

Plug in $s=0$ to get C.

$$F(0) = \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-x^2} e^{-i \cdot 0 \cdot x} \, dx}_{= \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}} = \frac{1}{\sqrt{2}} = C$$

So $\quad \mathcal{F}[e^{-x^2}] = \frac{1}{\sqrt{2}} e^{-s^2/4}$

engineers: s angular frequencies

Books: $\quad \mathcal{F}[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx} \, dx$

engineers: $2\pi s$ Hertz

Stein: $\quad \mathcal{F}[f] = \int_{-\infty}^{\infty} f(x) e^{-i \, 2\pi s x} \, dx$

Stein's choice makes many formulas cleaner.

Especially, $\mathcal{F}\left[e^{-\pi x^2}\right] = e^{-\pi s^2}$

Plancherel identity: $\|f\|^2 = \|\hat{f}\|^2$