Lesson 32 Fourier transform on Schwartz functions

Practice problems for Exam 2 on home page

google "Fourier transform", see Wikipedia page

1) $\mathcal{F}_1[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi s x}\, dx$ | $\mathcal{F}_1^{-1} = \int_{-\infty}^{\infty} \hat{f}_1(s)\, e^{i2\pi s x}\, ds$

2) $\mathcal{F}_2[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$ | $\mathcal{F}_2^{-1} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}_2(s)\, e^{isx}\, ds$

3) $\mathcal{F}_3[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$ | $\mathcal{F}_3^{-1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}_3(s)\, e^{isx}\, ds$

Remarks: $\mathcal{F}_1$ is nice for the theory : $\mathcal{F}_1[e^{-\pi x^2}] = e^{-\pi s^2}$

$\|f\|^2 = \|\hat{f}\|^2$ Plancherel identity nice!

$\mathcal{F}_2$ is nice for PDE solving because $\mathcal{F}_2[f'] = is\hat{f}(s)$

For practice problems and Exam 2, use $\mathcal{F}_2$.

Schwarz ⟸ German

Schwartz ⟸ French, theory of tempered distributions

Def^n: $\mathcal{S}$ is the space of C^∞-smooth fcns on $\mathbb{R}$ that are "rapidly decreasing at infinity", meaning that

$$\lim_{\substack{x\to\infty \\ x\to-\infty}} |f^{(n)}(x)|\,|x|^m = 0$$

$n, m = 0, 1, 2, \ldots$

<u>Exercise</u> $f \in \mathcal{S} \iff f$ C^∞-smooth and

$$\sup_{x \in \mathbb{R}} |f^{(n)}(x)|\,|x|^m < \infty$$

for $n, m = 0, 1, 2, \ldots$

Stein: p. 134–144 does everything! Fourier inversion

<u>Easy facts</u> $f \in \mathcal{S} \implies f' \in \mathcal{S}$

$f \in \mathcal{S} \implies x f(x) \in \mathcal{S}$

So $f \in \mathcal{S} \implies (\text{poly}) \cdot (\text{derivative of } f) \in \mathcal{S}$

<u>Big fact</u> $f \in \mathcal{S} \implies \hat{f} \in \mathcal{S}$

<u>For theory days:</u> $\hat{f}(s) = \displaystyle\int_{-\infty}^{\infty} f(x) e^{-i 2\pi s x}\, dx$

Fourier inversion: $f(x) = \displaystyle\int_{-\infty}^{\infty} \hat{f}(s) e^{i 2\pi s x}\, ds$

<u>Important properties of F.T. on $\mathcal{S}$</u> $\hat{f} = \mathcal{F}[f]$

1) $f(x+h) \xrightarrow{\mathcal{F}} e^{2\pi i s h} \hat{f}(s)$

2) $f(x) e^{-i 2\pi x h} \longrightarrow \hat{f}(s+h)$

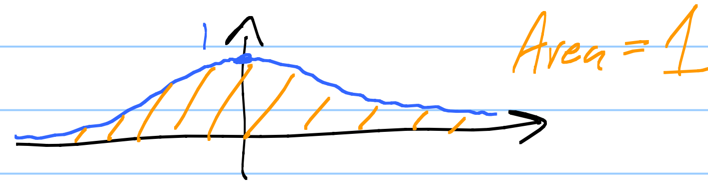
3) $f(\delta x) \longrightarrow \frac{1}{\delta} \hat{f}(s/\delta)$

4) $f'(x) \longrightarrow 2\pi i s \hat{f}(s)$

5) $-2\pi i\, x f(x) \longrightarrow \hat{f}\,'(s)$

Use these facts to show $\mathcal{F}\left[e^{-\pi x^2}\right] = e^{-\pi s^2}$

Write $\varphi(x) = e^{-\pi x^2}$



$$F(s) = \hat{\varphi}(s) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-i2\pi s x}\, dx$$

$$F'(s) = \int_{-\infty}^{\infty} e^{-\pi x^2} \left[\underbrace{-i2\pi x\, e^{-i2\pi s x}}_{\frac{\partial}{\partial s} e^{-i2\pi s x}} \right] dx$$

$$= i \int_{-\infty}^{\infty} \underbrace{\left[-2\pi x\, e^{-\pi x^2} \right]}_{\varphi'(x)} e^{-i2\pi s x}\, dx$$

$$= i\, \mathcal{F}[\varphi'] = i \left(i2\pi s\, \hat{\varphi}(s) \right)$$

$$= -2\pi s\, F(s)$$ Simple ODE

$$\boxed{F'(s) = -2\pi s\, F(s)}$$

$$\frac{F'(s)}{F(s)} = -2\pi s$$

$$\frac{d}{ds} \ln|F(s)| = -2\pi s$$

$$\ln|F(s)| = -\pi s^2 + C$$

$$|F(s)| = e^{\ln|F(s)|} = e^{-\pi s^2 + C} = e^{C} e^{-\pi s^2}$$

$$F(s) = \underbrace{\pm e^{C}}_{K} e^{-\pi s^2}$$

So $\quad F(s) = K e^{-\pi s^2}$

Get K from plugging in $s=0$.

$$F(0) = K e^{0} = K = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-i 2\pi \cdot 0 \cdot x}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1$$

So $\hat{\varphi} = \varphi$

Philosophy: Fourier inversion formula true on Gaussian "bump fcns" $\Longrightarrow$ true more generally.

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(x/\varepsilon)$$

Good kernel properties show $\varphi_\varepsilon * f \longrightarrow f$ nicely, especially for $f \in \mathcal{S}$!

$$(\varphi_\varepsilon * f)(x) = \underbrace{\int_{-\infty}^{\infty} \varphi_\varepsilon(x-t)\, f(t)\, dt}_{\approx \text{ Riemann sum}}$$

$$f \approx \underbrace{\sum}_{\text{Finite sum}} c_i \underbrace{\varphi_\varepsilon(x-t_i)}_{\text{bump fcn centered at } t_i}$$

Expanded f in Gaussian bump fcns.

Fourier inversion true on bumps $\Rightarrow$ true for $f \in \mathcal{S}$.

Later: F. inversion true on $\mathcal{S} \Rightarrow$ true $\mathcal{M}(\mathbb{R})$.

Philosophy Facts for $\mathcal{S} \Rightarrow$ facts for $L^2(\mathbb{R})$

How: $\mathcal{S}$ is dense in L^2.

EX: $\|f\|^2 = \|\hat{f}\|^2$ for $f \in \mathcal{S}$.

Given g in $L^2(\mathbb{R})$, get seq $f_n \in \mathcal{S}$ with

$$f_n \xrightarrow{L^2} g$$

$$\|f_n\|^2 = \|\hat{f}_n\|^2$$

$$\downarrow \qquad \downarrow$$

$$\|g\|^2 = \|\hat{g}\|^2$$ $\leftarrow$ Also see $\hat{g}$ makes sense as limit of $\hat{f}_n$ in L^2

Formula sheet for PDE version of $\tilde{\mathcal{F}}$ on home page.