Lesson 34 Fourier transform on $L^2(\mathbb{R})$. Jumps

Exam 2 released at 1:00 pm on Wednesday in Gradescope and on the home page. Due in Gradescope at 11:00 pm next Monday 4/17

Extending $\mathcal{F}: \mathcal{S} \to \mathcal{S}$ to $L^2(\mathbb{R})$

Step 1: Given $f \in L^2(\mathbb{R})$, approximate f in L^2 by something with compact support. Easy!

Use $\chi_{[-N,N]} \cdot f$ where $\chi_{[-N,N]} = \begin{cases} 1 & \text{in } [-N,N] \\ 0 & \text{outside} \end{cases}$

$\chi_{[-N,N]} \cdot f \to f$ in L^2 as $N \to \infty$ because

$\int_{|x|>N} |g|^2\, dx \to 0$ as $N \to \infty$ when $g \in L^2$.

Step 2: Let $f_n = \chi_{[-n,n]} \cdot f$.

Next, approximate f_n by a fcn in $C_0^\infty(\mathbb{R})$.

Bump fcn trick: Need bump fcn $\psi \in C^\infty(\mathbb{R})$ with support in $(-1,1)$. Get "approximation to identity" $\psi_\varepsilon(x) = \frac{1}{\varepsilon}\psi(x/\varepsilon)$.

Key: $\psi_\varepsilon * f_n$ is C^∞-smooth and it has compact support in $(-n-\varepsilon, n+\varepsilon)$.

Step 3: Young's inequality shows that

$$\Psi_\varepsilon * f_n \xrightarrow{L^2} f_n \quad \text{as } \varepsilon \to 0.$$

We now know: There is a sequence $\tilde{f}_n \in C_0^\infty(\mathbb{R})$ with $\tilde{f}_n \xrightarrow{L^2} f$.

Note: $C_0^\infty(\mathbb{R}) \subset \mathcal{S}$

Question: Why didn't we use C_0^∞ instead of $\mathcal{S}$?

Because $e^{-\pi x^2} \in \mathcal{S}$, but it doesn't have compact support!

How to extend $\mathcal{F}$ to L^2: Get $f_n \in C_0^\infty(\mathbb{R}) \subset \mathcal{S}$

$f_n \xrightarrow{L^2} f$. Plancherel identity: $\|f_n\|^2 = \|\hat{f}_n\|^2$

$f_n \xrightarrow{L^2} f \implies f_n$ is a Cauchy seq in L^2:

Given $\varepsilon > 0$, $\exists N$ such that $\|f_n - f_m\| < \varepsilon$ when $n, m > N$.

Aha! $\|f_n - f_m\| = \|\hat{f}_n - \hat{f}_m\|$

So $\hat{f}_n$ are a Cauchy seq too!

Big fact: $L^2(\mathbb{R})$ with Lebesgue integral is a complete Hilbert space, meaning every Cauchy seq converges in the L^2-norm.

So $\hat{f}_n$ converge in $L^2(\mathbb{R})$ to some fcn F.

Def^n: Can show F is uniquely determined by this method. Define: $\hat{f} = F$.

Fact: Plancherel identity holds on $L^2(\mathbb{R})$.

$$\tilde{\mathcal{F}} : L^2(\mathbb{R}) \xrightarrow[\text{onto}]{1-1} L^2(\mathbb{R}) \quad \text{isometric isomorphism.}$$

Simple fact: Inner product must be preserved too:

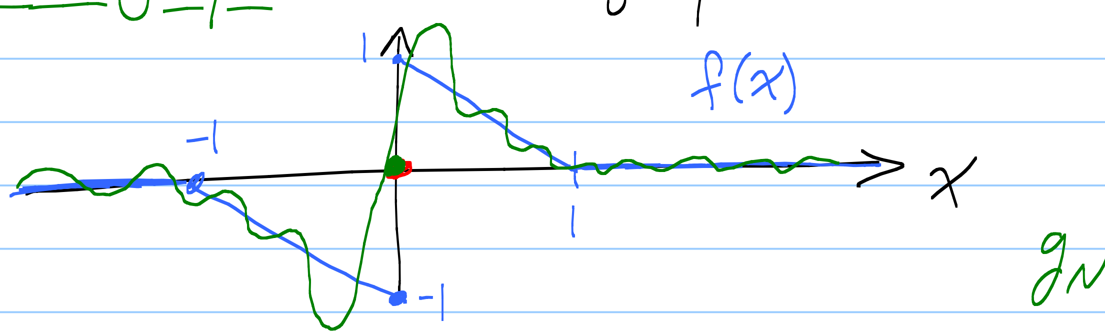
$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$$

Remark: $\mathbb{Q}$ is *not* complete. $\mathbb{R}$ is.

$\mathcal{R}$ = Riemann integrable L^2 fcns is *not* complete.

$\mathcal{R} \subset L^2(\mathbb{R}) \leftarrow$ complete!

What about jumps? Model jump



Easy to calculate $\hat{f}(0) = 0 \leftarrow$ midpoint of jump!

$$\hat{f}(s) = \lim_{N\to\infty} \underbrace{\int_{-N}^{N} f(x)\, e^{-i2\pi s x}\, dx}_{g_N}$$

Subtract c(Model jump) from something with a jump to get rid of jump. Use translation identities to see facts about jumps at pts $x \neq 0$.

Prac. prob 1: $\left| \int_{|x|>N} g(x) \sin wx \, dx \right| \leq \int_{|x|>N} |g(x)| \underbrace{|\sin wx|}_{\leq 1} dx$

$$\leq \int_{|x|>N} |g(x)|\, dx \longrightarrow 0 \quad \text{as } N\to\infty$$

because $\underbrace{\int_{-\infty}^{\infty} |g(x)|\, dx}_{I} = \underbrace{\int_{-N}^{N} |g(x)|\, dx}_{\to I \text{ as } N\to\infty} + \underbrace{\int_{|x|>N} |g|\, dx}_{\text{must} \to 0 \text{ as } N\to\infty,}$

Prob 2: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_0^1 \underbrace{x}_{u}\, \underbrace{e^{-isx}\, dx}_{dv}$$

$du = dx$

$v = \frac{1}{-is} e^{-isx}$

Int. by parts!

$\underline{\underline{or}}$ $$= \frac{1}{\sqrt{2\pi}} \int_0^1 \underbrace{x}_{u} \left(\cos(sx) - i\sin(sx) \right) dx$$

F_s integral

$\underline{\text{Prob 3}}$: Fourier sine transf : Lesson 27

$\underline{\text{Prob 4}}$: Complex F.T. : Lesson 28

Ask yourself: Why is the end result a real valued sol^n?