Lecture 36 Fundamental Theorem of Algebra (cont'd)

Exam 2 due in GS tonight 11:00 pm

Complex rational function $R(z) = \frac{P(z)}{Q(z)}$ polys

$$R(x+iy) = u(x,y) + iv(x,y)$$

$\boxed{u, v \text{ harmonic away from zeroes of } Q(z).}$

Lemma If $P(z)$ is a complex polynomial of deg $N \geq 1$, then there are constants $0 < A < B$ and a radius $R_0 > 0$ such that

$$A|z|^N \leq |P(z)| \leq B|z|^N.$$

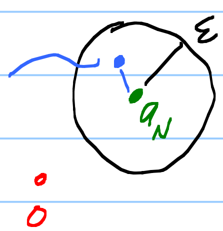
[So $|P(z)| \to \infty$ as $|z| \to \infty$.]

Pf:

$$\frac{P(z)}{z^N} = \frac{a_N z^N + \cdots + a_1 z + a_0}{z^N} \qquad (a_N \neq 0)$$

$$= a_N + \underbrace{\left(\frac{a_{N-1}}{z} + \frac{a_{N-2}}{z^2} + \cdots + \frac{a_1}{z^{N-1}} + \frac{a_0}{z^N}\right)}_{|\text{each term}| \to 0 \text{ as } |z| \to \infty.}$$

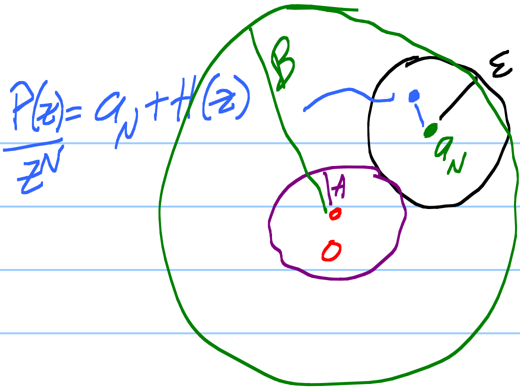
$$\frac{P(z)}{z^N} = a_N + H(z)$$



$$\frac{P(z)}{z^N} = a_N + \underbrace{(\text{small})}_{H(z)}$$

$\varepsilon < |a_N|$

Want $|H(z)| < \varepsilon$ when $|z| < R_0$.



$$A < \left|\frac{P(z)}{z^N}\right| < B$$

when $|z| > R_0$.

Mult by $|z|^N$ to get

Basic polynomial ineq : $A|z|^N < |P(z)| < B|z|^N$

Hmmm : Suppose $P(z)$ is a poly of deg $N \geq 1$ that does not have a root in $\mathbb{C}$.

Basic est : $A|z|^N \leq |P(z)| \leq B|z|^N$ when $|z| > R_0$.

Let $R(z) = \frac{1}{P(z)}$ ← $\mathbb{C}$-diff on $\mathbb{C}$

So $R(x+iy) = u(x,y) + iv(x,y)$

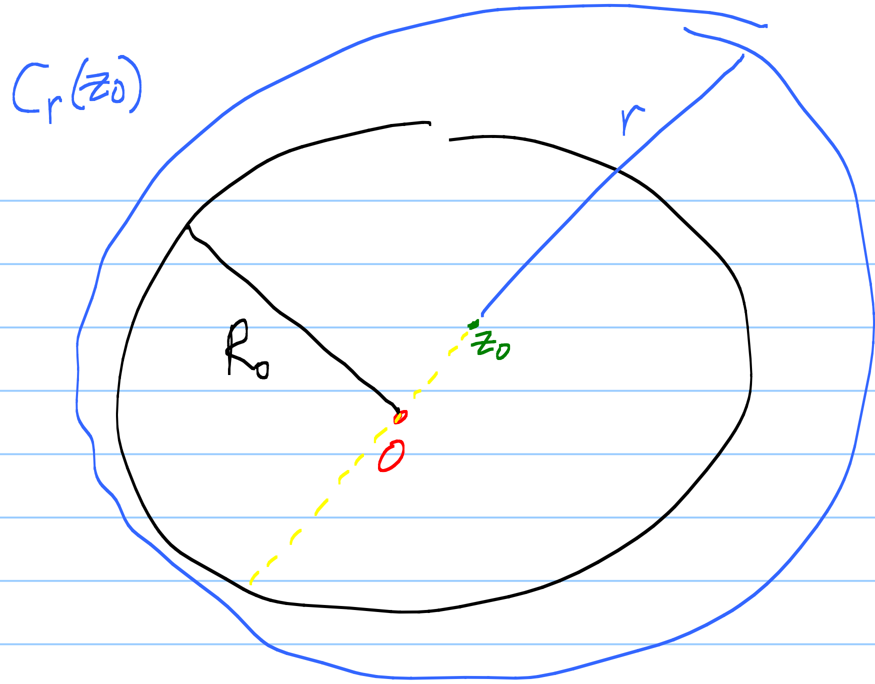
harmonic on $\mathbb{C}$

and $|R(z)| = \frac{1}{|P(z)|} \leq \frac{1}{B|z|^N}$ when $|z| > R_0$.

Aha! $|R(z)| \to 0$ as $|z| \to \infty$.

Claim: Ave prop for harmonic fcns shows $u \equiv 0, v \equiv 0$.

Pf: Pick $z_0 = x_0 + iy_0 \in \mathbb{C}$



$\left|R(z)\right| < \varepsilon$

if $|z| > R_0$

Need $r > R_0 + |z_0|$.

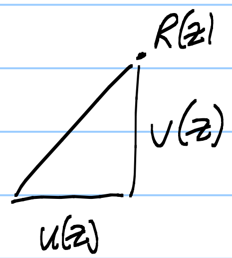
$C_r(z_0)$ outside

$\overline{D_R(z_0)}$.

Aha! Ave property shows

$$u(x_0,y_0) = \text{Ave of } u \text{ on } C_r(z_0) \le \max_{C_r(z_0)} |u|$$

$$\le \max_{C_r(z_0)} |R(z)| < \varepsilon$$



Same for $v(x_0, y_0)$. Since ε arbitrary, get

$u(x_0,y_0)$, $v(x_0,y_0) \le 0$. x_0, y_0 arbitrary too!

So $u, v \le 0$.

Repeat using $-R(z)$ in place of $R(z)$ to see

$u, v \ge 0$. So $u \equiv 0$, $v \equiv 0$.

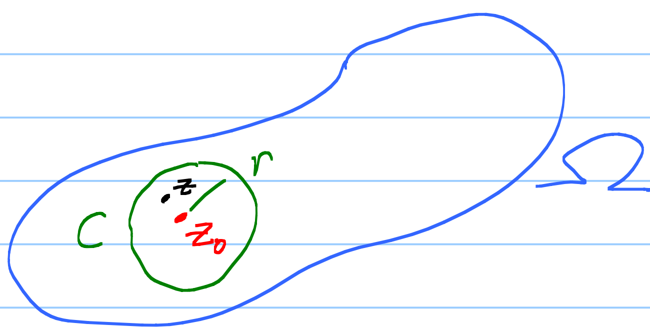
Contradiction: $|R(z)| = \underbrace{\frac{1}{|P(z)|}}_{\text{never zero}} = \frac{1}{\sqrt{u^2+v^2}}$

P must have a root!

History: Easy to see that rational fcns $R(z)$ were $\mathbb{C}$-diff because freshman calc facts hold.

MA 425: $\mathbb{C}$-diff'ble fcns on an open set Ω.
= {Analytic fcns on Ω}.

Big fact:



$\overline{D_r(z_0)} \subset \Omega$

Cauchy integral formula: $f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z}\, dw$

Consequently: 1) Max princ for $|f|$ holds

2) f is infinitely complex diff'ble, and so $\operatorname{Re} f$ and $\operatorname{Im} f$ are harmonic!

3) f = convergent power series on $D_r(z_0)$.

History: Liouville's theorem. A bounded entire fcn must be constant.

[$f(z)$ entire: f is $\mathbb{C}$-diff'ble on $\mathbb{C}$

bounded: $|f(z)| < M$ on $\mathbb{C}$]

Simple pf of Fund Thm Alg:

$P(z)$ no roots, basic poly est $\Rightarrow$ $\frac{1}{P(z)}$ is a bounded entire fcn. Liouville's $\Rightarrow$ $\frac{1}{P(z)} = \text{const.}$ $\lightning$