Review 2

Final Exam Wed, May 3, 7-9 pm, Schm 313
$\overset{\shortparallel}{\text{REC}}$

p. 91; 14. Show that if f is 2π-periodic on $\mathbb{R}$ and C^1-smooth on $\mathbb{R}$, then the F.S. $S_N(x) = \sum_{-N}^{N} \underbrace{\hat{f}(n)}_{c_n} e^{inx}$ converges absolutely and uniformly to f.

Key: Parseval's, $\hat{f'}(n) = -in\hat{f}(n)$, Cauchy-Schwarz ineq, Weierstraß M-test.

Why: $\hat{f'}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}\,dx$

Pf: Integrate by parts $\begin{cases} u = e^{-inx} \\ dv = f'(x)\,dx \end{cases}$

How to remember: $f \sim \sum_{-\infty}^{\infty} \hat{f}(n)e^{-inx}$

Expect $f' \sim \frac{d}{dx}$ of this

$$\sim \sum_{-\infty}^{\infty} \underbrace{-in\hat{f}(n)}_{\text{expect } = \hat{f'}(n)} e^{-inx}$$

Parseval's: $\sum_{-\infty}^{\infty} |\hat{g}(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x)|^2\,dx$

(Holds even when g is merely piecewise cont.)

So $$\sum_{-\infty}^{\infty} |\underbrace{-in\hat{f}(n)}_{\hat{f'}(n)}|^2 = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(x)|^2\,dx}_{< \infty \text{ because } f' \text{ cont.}}$$

$$\boxed{\sum_{-N}^{N} n^2 \left|\hat{f}(n)\right|^2 < M}$$

Hmmm: $\underline{\displaystyle\sum_{-N}^{N} \left|\hat{f}(n)\right|} = \displaystyle\sum_{-N}^{N} \frac{1}{|n|} \cdot \underbrace{|n| \left|\hat{f}(n)\right|}$

$$\vec{u} = (u_1, u_2, \ldots, u_N)$$

$$\vec{v} = (v_1, v_2, \ldots, v_N)$$

Cauchy-Schwarz $\quad |\vec{u}\cdot\vec{v}| \le |\vec{u}|\,|\vec{v}|$

$$\left|\sum_{1}^{N} u_n \bar{v}_n\right| \le \left(\sum |u_n|^2\right)^{1/2} \left(\sum |v_n|^2\right)^{1/2}$$

$\displaystyle\sum_{1}^{N}$ Half: $\quad \left|\left(\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{N}\right)\cdot\left(1\cdot\left|\hat{f}(1)\right|, \ldots, N\left|\hat{f}(N)\right|\right)\right|$

$$\le \underbrace{\left(\sum_{1}^{N} \frac{1}{n^2}\right)^{1/2}}_{< \left(\frac{\pi^2}{6}\right)^{1/2}} \underbrace{\left(\sum_{1}^{N} n^2 \left|\hat{f}(n)\right|^2\right)^{1/2}}_{< M^{1/2}}$$

$\displaystyle\sum_{-N}^{-1}$ Half: $\quad$ same estimate!

So $\quad \displaystyle\sum_{-N}^{N} \left|\hat{f}(n)\right| < 2\cdot\frac{\pi}{\sqrt{6}}\cdot M^{1/2} + \Big|\underbrace{\hat{f}(0)}_{\text{ave of } f \text{ on } [-\pi,\pi]}\Big|$

Finite!

$\left|\hat{f}(n)\right| = \left|\hat{f}(n)e^{-inx}\right|$. $\quad$ Weierstraß M-test: $M_n = \left|\hat{f}(n)\right|$.

$$\sum M_n < \infty.$$

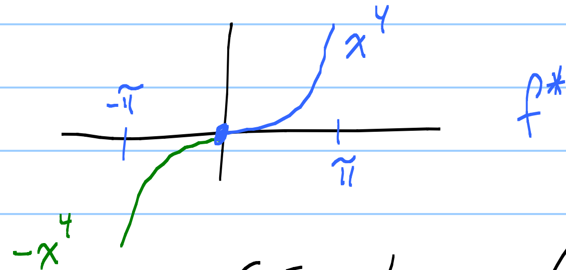
2. If the Fourier sine Series of $f(x) = x^4$ on $[0, \pi]$ is $\sum_{n=1}^{\infty} b_n \sin nx$, evaluate the sum of the infinite series $\sum_{n=1}^{\infty} b_n^2$.

Parseval's: f piecewise cont on $[-\pi, \pi]$

$$f \sim a_0 + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx)$$

Then $$2a_0^2 + \sum_1^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} |f|^2 \, dx$$

Aha! Extend $f(x) = x^4$ on $[0, \pi]$ to $[-\pi, \pi]$ as an odd fcn.



Cosine terms A_n all zero.

Fourier coeff for f^*:

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\underbrace{f^*(x)}_{\text{odd}} \underbrace{\sin nx}_{\text{odd}}}_{\text{even}} \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \underbrace{f^*(x)}_{=f(x)} \sin nx \, dx = b_n$$

Parseval's: $$2 \cdot 0^2 + \sum_1^{\infty} (0^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} |f^*(x)|^2 \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (x^4)^2 \, dx$$

$$= \frac{2}{\pi} \frac{1}{9} x^9 \Big|_0^{\pi} = \frac{2}{9} \pi^8 .$$

3. Suppose $f(x) = 1 + \cos x + \sum_{n=1}^{\infty} \frac{1}{n^2} \sin nx$ on $[-\pi, \pi]$.

Explain why f is continuous.

Evaluate $\int_{-\pi}^{\pi} f(x) \cos 5x \, dx$ and $\int_{-\pi}^{\pi} f(x) \sin 5x \, dx$.

Key: W M-test. Suppose $f_n(x)$ are continuous on $[a,b]$ and $|f_n(x)| \leq M_n$ with $\sum_1^\infty M_n < \infty$. Then $S_N(x) = \sum_1^N f_n(x)$ converge absolutely and uniformly on $[a,b]$ to a continuous fcn $f(x)$.

$$\left| \frac{1}{n^2} \sin nx \right| \leq \frac{1}{n^2} \underbrace{|\sin nx|}_{\leq 1} \leq \frac{1}{n^2} \leftarrow M_n$$

$$\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6} < \infty. \quad \text{Done!}$$

2nd half: Keys: 1) Uniform conv $\Rightarrow$ $\int \sum = \sum \int$

2) 1, $\cos nx$, $\sin nx$ $\perp$ on $[-\pi,\pi]$

So $\int_{-\pi}^{\pi} f(x) \cos 5x \, dx = 0$

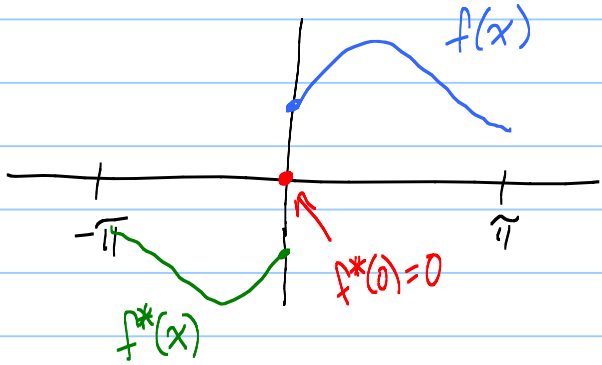
$$\int_{-\pi}^{\pi} f(x) \sin 5x \, dx = \int_{-\pi}^{\pi} \frac{1}{5^2} \sin 5x \cdot \sin 5x \, dx$$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$$= \frac{1}{25} \underbrace{\int_{-\pi}^{\pi} \sin^2 5x \, dx}_{\pi/2}$$

6. Suppose that f is continuous on $[0, \pi]$ and that $\int_0^\pi f(x) \sin nx \, dx = 0$ for every positive integer n. Explain why f must be identically zero.

Aha! Let f^* be the <u>odd</u> extension of f to $[-\pi, \pi]$



f^* is piecewise cont with a possible jump at $x=0$.

f^* odd : a_0, a_n's all zero.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f^*(x) \sin nx \, dx = \underbrace{\frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx}_{=0}$$

Parseval's for f^* : $0 = 0^2 + \sum(0^2+0^2) = \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} |f^*|^2 \, dx}$

f^* piecewise cont

$\int = 0 \Rightarrow f^*(x) = 0$

at pts x where f^* is continuous.

So $f(x) = 0, \quad 0 < x < \pi$.

Aha! f cont on $[0, \pi] \Rightarrow f(0) = 0, f(\pi) = 0$ too.

5. Suppose that $u(r,\theta)$ is the harmonic function with continuous boundary values on the unit circle given in polar coordinates by

$$u(1,\theta) = \sin\theta + \cos 2\theta. \leftarrow f(\theta)$$

Evaluate $u\left(\frac{1}{2}, \frac{\pi}{2}\right)$.

Key: Solving D. Prob on $D_1(0)$: $r^n \cos n\theta$, $r^n \sin n\theta$ are harmonic. Try $a_0 + \sum_{n=1}^{\infty} \left(a_n r^n \cos n\theta + b_n r^n \sin n\theta\right)$

$r=1$: See a_n's, b_n's are Fourier coeff for bndry value fcn $f(\theta)$.

Rather than calculating $\int$'s, most of which $= 0$:

$u(r,\theta) = r^1 \sin\theta + r^2 \cos 2\theta$ is harmonic on $D_1(0)$. And

$u(1,\theta) = \sin\theta + \cos 2\theta$ ✓

Note: $g(\theta) = 7 + \sin 27\theta + \cos 13\theta$

ans: $7 + r^{27} \sin 27\theta + r^{13} \cos 13\theta$

Friday: Fourier transform. Answer questions.