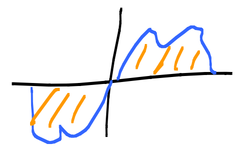
Review for Exam 1

Exam 1 in class Wed.
See home page for info.
HWK 4 solutions posted
on home page now.

1. Explain why

Symmetric
$$\int_{-\pi}^{\pi} \underbrace{\overbrace{e^{-x^2}}^{\text{even}} \overbrace{\sin 3x}^{\text{odd}}}_{\text{odd}} \, dx = 0$$



without trying to find an antiderivative (which is impossible).

2. Find all possible *positive* values of λ that allow non-zero solutions to the boundary value problem

$$X''(x) + \lambda X(x) = 0$$

on $[0, \pi]$ with $X(0) = 0$ and $X'(\pi) = 0$. For each such λ, find a non-zero solution.

$r^2 + \lambda = 0$

$r = \pm i\sqrt{\lambda}$

$u(0,t) = 0$ refrigerated left

$\frac{\partial u}{\partial x}(\pi, t) = 0$ insulated right side

Kelvin's law: (rate heat flow) $= -c$ (temp gradient) $\frac{\partial u}{\partial x}$

Write $\lambda = k^2$. So $\sqrt{\lambda} = k$.

$$X(x) = c_1 \cos kx + c_2 \sin kx$$

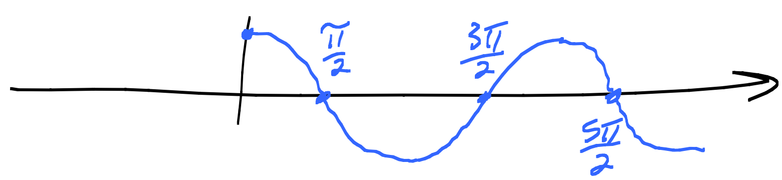
Need $X(0) = c_1 \cdot 1 + c_2 \cdot 0 \overset{\text{want}}{=} 0$ $\qquad \boxed{c_1 = 0}$

$$X'(x) = 0 + c_2 k \cos kx$$

Need $X'(\pi) = c_2 k \cos k\pi \overset{\text{want}}{=} 0$

Don't want $c_2 = 0$!

Need $\cos k\pi = 0$



Need $k\pi = (\text{odd})\frac{\pi}{2} = (2n-1)\frac{\pi}{2}$, $n = 1, 2, 3, \ldots$

Ans: $X(x) = c \sin \frac{2n-1}{2} x$

$k = \frac{2n-1}{2}$ $\quad n = 1, 2, \ldots$

$\lambda = k^2 = \left(\frac{2n-1}{2}\right)^2$, $n = 1, 2, \ldots$

3. Given that

$$1+z+z^2+\cdots+z^N=\frac{1-z^{N+1}}{1-z}$$

$1-z+z^2+\cdots+(-1)^N z^N$

$= \frac{1-(-1)^{N+1}z^{N+1}}{1+z}$

when z is a complex number unequal to one, find a short closed expression for

$$1-\sin\theta+\sin 2\theta-\sin 3\theta+\cdots+(-1)^N\sin N\theta$$

$z=e^{i\theta}$

$z^n=e^{in\theta}$

that does not involve a sum of length N. (Do not try to simplify the expression you get.)

Hints: What happens if you replace z by $-z$? Note that $\sin n\theta$ is the imaginary part of $e^{in\theta}$.

$$1-\mathrm{Im}\left[e^{i\theta}-e^{i2\theta}+e^{i3\theta}+\cdots+(-1)^{N-1}e^{iN\theta}\right]$$

$$1-\mathrm{Im}\left[z\left(1-z+z^2+\cdots+(-1)^{N-1}z^{(N-1)}\right)\right]$$

$$1-\mathrm{Im}\left[\frac{z\left(1-(-1)^N z^N\right)}{1+z}\right]$$

$$1-\mathrm{Im}\left[\underbrace{\frac{e^{i\theta}\left(1-(-1)^N e^{iN\theta}\right)}{1+e^{i\theta}}}_{\frac{\left(\cos\theta-(-1)^N\cos(N+1)\theta\right)+i\left(\sin\theta-(-1)^N\sin(N+1)\theta\right)}{(1+\cos\theta)+i\sin\theta}}\right]$$

or mult by $\frac{e^{-i\theta/2}}{e^{-i\theta/2}}$

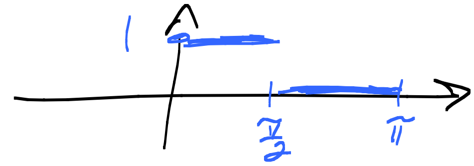
$$\frac{A+Bi}{C+Di}\cdot\frac{C-Di}{C-Di}$$

$$\mathrm{Im}=\frac{BC-AD}{C^2+D^2} \longrightarrow \frac{(AC+BD)+i(BC-AD)}{C^2+D^2}$$

4. Assume that

2π-periodic

$$S(x) = \sum_{n=1}^{\infty} B_n \underbrace{\sin nx}_{\text{odd}}$$



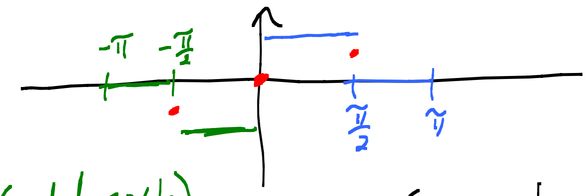
is the sum of the Fourier sine series on $[0, \pi]$ for the function $f(x)$ that is equal to one on the interval $[0, \pi/2)$ and zero on $[\pi/2, \pi]$.

a) Write an integral formula for B_n and evaluate the integral (but don't simplify it or try to see a pattern). $B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi/2} \sin nx\,dx$ = easy

b) Graph the function that the Fourier sine series converges to on $[-3\pi, 3\pi]$, being careful at jumps.

c) Evaluate the sum

$$\sum_{n=1}^{\infty} {B_n}^2.$$



(odd ext)

(see below)

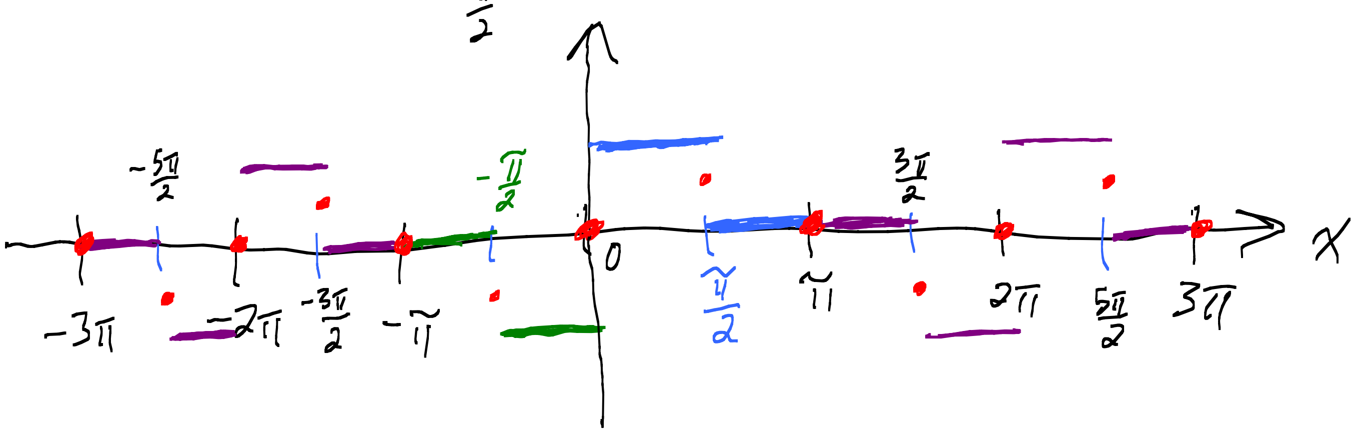
Hint: The Fourier sine series is equal to the full Fourier series of the function you graphed in part (b).

Parseval's

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)^2}_{\substack{\text{odd}\\ \text{extension}\\ \text{to } [-\pi,\pi]}}\,dx = 2\underset{\uparrow\atop 0}{a_0^2} + \sum_{n=1}^{\infty}(\underset{\uparrow\atop 0}{a_n^2} + b_n^2)$$

$$\underbrace{}$$

$$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\pi/2} 1^2\,dx$$



2π-periodic odd extension

1. Suppose that $f(x)$ is a C^1-smooth function on $[a,b]$. Use integration by parts to show that

$$I = \int_a^b \underbrace{f(x)}_{u} \underbrace{\cos(Nx)\,dx}_{dv}$$

$v = \frac{1}{N} \sin Nx \, dx$

$du = f' dx$

tends to zero as the positive integer N tends to infinity.

2. Estimate the integral in problem one in terms of the maximum value M of $|f(x)|$ on $[a,b]$.

$$|I| \leq \left| \int_a^b \right| \leq \int_a^b \underbrace{|f(x)|}_{\leq M} \underbrace{|\cos Nx|}_{\leq 1} dx$$

$$\leq M \cdot 1 \cdot (b-a)$$

3. Show that the integral in problem one goes to zero as $N \to \infty$ when f is merely assumed to be continuous by approximating f by a polynomial and then using your results from problem one and two.

$$|f(x) - P(x)| < \varepsilon$$

poly

$$\underbrace{\left| \int_a^b f(x) \cos Nx \, dx \right|}_{\text{Can make arbitrarily small. Must } = 0} = \left| \int_a^b (f(x) - P(x)) \cos Nx + P(x) \cos Nx \, dx \right|$$

#2

$$\leq \left| \int_a^b (f(x) - P(x)) \cos Nx \, dx \right| \leq \varepsilon \cdot (b-a)$$

$$+ \left| \int_a^b P(x) \cos Nx \, dx \right| \to 0 \quad \#1$$

4. Suppose $P(x)$ is a positive continuous function on $[0,1]$ such that $P(1/2) = 10^4$, and $P(x) < 1/100$ when x is in $[0,1/4] \cup [3/4,1]$, and

$$\int_0^1 P(x)\,dx = 1.$$

If f is a continuous function such that $|f(x)| < 17$ on $[0,1]$, but $|f(x)| < 1/3$ on $[1/4,3/4]$, estimate how large

$$\left|\int_0^1 P(x)f(x)\,dx\right| = \Big| \int_0^{1/4} + \int_{3/4}^{1} \quad P \text{ small}$$

could be. What is your best estimate for the bound if the assumption that P is positive is dropped?

$$+ \underbrace{\int_{1/4}^{3/4} \quad f \text{ small} \Big|}$$

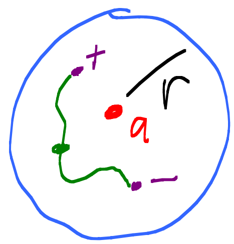
$$\left|\int_{1/4}^{3/4} P(x) f(x)\,dx\right| \le \int_{1/4}^{3/4} P(x) \underbrace{|f(x)|}_{<\frac{1}{3}} dx$$

$$< \frac{1}{3} \underbrace{\int_{1/4}^{3/4} P(x)\,dx}$$

$$< \underbrace{\int_0^1 P(x)\,dx}_{1}$$

5. Prove that harmonic functions cannot have isolated zeroes.

Ave property!



only zero of u in $D_r(a)$