MA 428 HWK 1 solutions

1. $\frac{\partial}{\partial x} f(x+ct) = f'(x+ct)\frac{\partial}{\partial x}[x+ct] = f'(x+ct)\cdot 1$

$$\frac{\partial^2}{\partial x^2} f(x+ct) = f''(x+ct)\cdot 1 \cdot 1$$

$$\frac{\partial}{\partial t} f(x+ct) = f'(x+ct)\frac{\partial}{\partial t}[x+ct] = f'(x+ct)\cdot c$$

$$\frac{\partial^2}{\partial t^2} f(x+ct) = f''(x+ct)\cdot c \cdot c$$

See that $u = f(x+ct)$ satisfiess

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}.$$

Replacing c by $-c$ above shows that $f(x-ct)$ does too (because $(-c)^2 = c^2$).

Since $L = \frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}$ is a linear operator, i.e.,

$L(c_1 u_1 + c_2 u_2) = c_1 L u_1 + c_2 L u_2$, it follows

that if u_j satisfy $Lu_j = 0$, $j = 1, 2$, then

so does $c_1 u_1 + c_2 u_2$.

2. Assume $u(x,t) = X(x)T(t)$ and plug into the PDE: Get $X(x)T''(t) = c^2 X''(x)T(t)$

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} = \lambda$$

$X'' - \lambda X = 0$ | $T'' - c^2\lambda T = 0$

B.C. $\begin{cases} X(0) = 0 \\ X(L) = 0 \end{cases}$

Case $\lambda = 0$: $X(x) = Ax + B$

Only the zero sol^n satisfies the B.C.

Case $\lambda > 0$: Write $\lambda = k^2$.

$$X(x) = c_1 e^{kx} + c_2 e^{-kx} = A_1 \cosh kx + A_2 \sinh kx$$

$$X(0) = A_1 = 0.$$

$$X(L) = 0 \cdot \cosh kL + A_2 \underbrace{\sinh kL}_{\text{not zero}} = 0$$

So $A_2 = 0$ too. Get only the zero sol^n.

<u>Case $\lambda < 0$</u>: Write $\lambda = -k^2$.

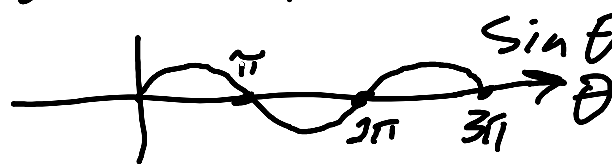
Get $X(x) = c_1 \cos kx + c_2 \sin kx$

$X(0) = c_1 = 0$. So $X(x) = c_2 \sin kx$

$$X(L) = c_2 \sin kL \overset{\text{want}}{=} 0$$

Don't want $c_2 = 0$ too, so need

$\boxed{\sin kL = 0}$



Need $kL = n\pi$, $n = 1, 2, 3, \ldots$

$\boxed{k = \frac{n\pi}{L}}$ $n = 1, 2, 3, \ldots$

$\lambda = -k^2 = -\left(\frac{n\pi}{L}\right)^2$. Take $c_2 = 1$.

Get $X_n(x) = \sin \frac{n\pi x}{L}$, $n = 1, 2, 3, \ldots$

T-problem: $\quad T'' + c^2\left(\frac{n\pi}{L}\right)^2 T = 0$

$$T_n(t) = A_n \sin\frac{cn\pi t}{L} + B_n \cos\frac{cn\pi t}{L}$$

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)\, T_n(t) =$$

$$= \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n \sin\frac{cn\pi t}{L} + B_n \cos\frac{cn\pi t}{L}\right)$$

Note that

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(\frac{cn\pi}{L} A_n \cos\frac{cn\pi t}{L} - \frac{cn\pi}{L} B_n \sin\frac{cn\pi t}{L}\right)$$

Finally the initial conditions yield

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \overset{\text{want}}{=} f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \frac{cn\pi}{L} A_n \sin\frac{n\pi x}{L} \overset{\text{want}}{=} g(x)$$

The functions $\sin\frac{n\pi x}{L}$ are orthogonal on $[0,L]$ and $\int_0^L \sin^2\frac{n\pi x}{L}\,dx =$

$$\int_0^L \frac{1}{2}\left(1-\cos\frac{2n\pi x}{L}\right)dx = \frac{L}{2}.$$

Hence, multiplying

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

by $\sin\frac{m\pi x}{L}$ and integrating $\int_0^L$ yields

$$\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx = B_m\cdot\frac{L}{2}.$$

So $\boxed{B_m = \frac{2}{L}\int_0^L f(x)\sin\frac{m\pi x}{2}\,dx}$

Similarly, if

$$g(x) = \sum_{n=1}^{\infty} \left(\frac{cn\pi}{L} A_n \right) \sin \frac{n\pi x}{L},$$

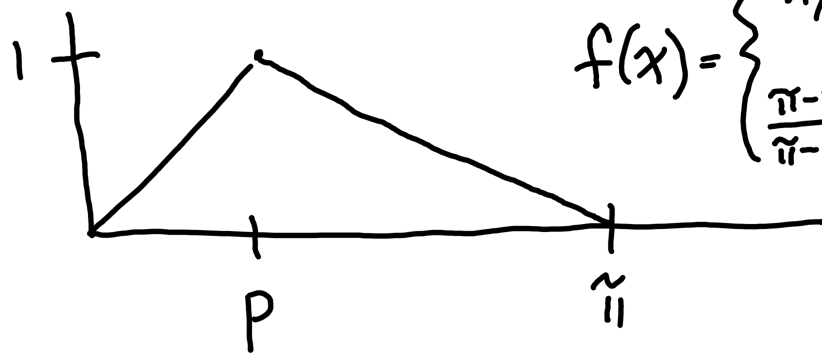
then $\frac{cn\pi}{L} A_n$ are the Fourier sine series coeff for g, i.e.,

$$\frac{cm\pi}{L} A_m = \frac{2}{L} \int_0^L g(x) \sin \frac{m\pi x}{L} \, dx$$

and we get

$$\boxed{A_m = \frac{2}{cm\pi} \int_0^L g(x) \sin \frac{m\pi x}{L} \, dx}.$$

3.



$$f(x) = \begin{cases} x/p & 0 \le x \le p \\ \dfrac{\pi - x}{\pi - p} & p \le x \le \pi \end{cases}$$

$$B_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{p} \frac{x}{p} \sin nx \, dx$$

$$+ \frac{2}{\pi} \int_p^{\pi} \frac{\pi - x}{\pi - p} \sin nx \, dx$$

= ... integration by parts ---

$$= \frac{2}{n^2} \frac{\sin np}{p(\pi - p)}$$

4. Even terms B_{2n} all zero when $(2n)p$ are all zeroes of $\sin x$, and this happens exactly when $p = \frac{\pi}{2}$.

[It happens when $p = \frac{\pi}{2}$. If it happens for p, $0 < p < \pi$, then $2p$ must be a zero of $\sin x$. But $0 < 2p < 2\pi$. So $2p = \pi$ and we see that $p = \frac{\pi}{2}$.]

$A_m = 0$ if mp is a root of $\sin x$, i.e., if $mp = n\pi$ for some $n = 1, 2, 3, \dots$. But $p = \frac{n}{m}\pi$ where $0 < p < \pi$ requires $n = 1, 2, \dots, m-1$. If A_3 and $A_5 = 0$, then $p \in \{\frac{\pi}{3}, \frac{2\pi}{3}\} \cap \{\frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}\} = \emptyset$.

Hence, there is no $p \in (0, \pi)$ making A_3, A_5 zero. So it's even harder to make $A_3, A_5, A_7, A_9, \ldots$ all zero.