MA 428 HWK 2 solutions

1. $$B_n = \frac{2}{\pi}\int_0^{\pi} f'(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \underbrace{\sin nx}_{u}\ \underbrace{f'(x)\,dx}_{dv}$$

$u = \sin nx \quad du = n\cos nx\, dx$

$dv = f'(x)dx \quad v = f(x)$

$$= \frac{2}{\pi}\left[\underbrace{uv\Big|_0^{\pi}}_{\substack{=0 \text{ because} \\ \sin 0 = 0,\ \sin n\pi = 0}} - \int_0^{\pi} v\, du\right]$$

$$= -\frac{2}{\pi}\int_0^{\pi} f(x)\ n\cos nx\, dx$$

$$= -n\, A_n \,, \quad A_n = \text{F. Cosine coeff for } f(x)$$

2. $\langle 1, x^7 \rangle = \int_{-\pi}^{\pi} \underbrace{1 \cdot x^7}_{\text{odd}} \, dx = \underline{\underline{0}}$

because $\int_{-A}^{A} \text{odd} \, dx = \underline{\underline{0}}$

$\langle 1, \cos 5x \rangle = \int_{-\pi}^{\pi} \cos 5x \, dx = \frac{1}{5} \sin 5x \Big|_{-\pi}^{\pi}$

$= 0 - 0 = 0$

$\langle x^7, \cos 5x \rangle = \int_{-\pi}^{\pi} \underbrace{\underbrace{x^7}_{\text{odd}} \; \underbrace{\cos 5x}_{\text{even}}}_{\text{odd}} \, dx = \underline{\underline{0}}$

$\langle 1, 1 \rangle = \int_{-\pi}^{\pi} 1 \cdot 1 \, dx = 2\pi$

$\langle x^7, x^7 \rangle = \int_{-\pi}^{\pi} x^{14} \, dx = \frac{1}{15} x^{15} \Big|_{-\pi}^{\pi} = \frac{2\pi^{15}}{15}$

$\langle \cos 5x, \cos 5x \rangle = \int_{-\pi}^{\pi} \frac{1}{2}(1 + \cos 10x) \, dx = \frac{1}{2} \cdot 2\pi = \pi$

$$\int_{-\pi}^{\pi} f(x)\cdot 1\,dx = \int_{-\pi}^{\pi} A + Bx^7 + C\cos 5x\,dx$$

$$= 2\pi A + 0 + 0$$

$$\boxed{A = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

$$\int_{-\pi}^{\pi} f(x)\cdot x^7\,dx = A\cdot 0 + B\langle x^7, x^7\rangle + C\cdot 0$$

$$= B\frac{2\pi^{15}}{15}$$

$$\boxed{B = \frac{15}{2\pi^{15}}\int_{-\pi}^{\pi} f(x)\,x^7\,dx}$$

$$\int_{-\pi}^{\pi} f(x)\cos 5x\,dx = A\cdot 0 + B\cdot 0 + C\cdot\pi$$

$$\boxed{C = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 5x\,dx}$$

$$\int_{-\pi}^{\pi} f(x)^2\,dx = \int_{-\pi}^{\pi} (A + Bx^7 + C\cos 5x)^2\,dx$$

$$= \int_{-\pi}^{\pi} A^2 + B^2(x^7)^2 + C^2\cos^2 5x$$

$$+ AB\cdot 1\cdot x^7 + AC\cdot 1\cdot \cos 5x + BCx^7\cos 5x\,dx$$

$$= A^2\cdot 2\pi + B^2\cdot \frac{2\pi^{15}}{15} + C^2\cdot \pi$$

$$+ AB\cdot 0 + AC\cdot 0 + BC\cdot 0$$

$$= \underline{\underline{2\pi A^2 + \frac{2\pi^{15}}{15}B^2 + \pi C^2}}$$

3. $\sin x + \sin 2x + \cdots + \sin Nx$

$$= \operatorname{Im}\left[e^{ix} + e^{i2x} + \cdots + e^{iNx} \right]$$

$$= \operatorname{Im}\left[(e^{ix}) + (e^{ix})^2 + \cdots + (e^{ix})^N \right] =$$

$$= \mathrm{Im}\left[e^{ix} (1 + e^{ix} + \cdots + e^{i(N-1)x} \right]$$

$$= \mathrm{Im}\left[e^{ix} \left(\frac{1 - e^{iNx}}{1 - e^{ix}} \right) \cdot \underbrace{\left(\frac{-e^{-ix/2}/2i}{-e^{-ix}/2i} \right)}_{=1} \right]$$

$$= \mathrm{Im}\left[\frac{\frac{i}{2} e^{x/2} - \frac{i}{2} e^{i(N+\frac{1}{2})x}}{\sin \frac{x}{2}} \right]$$

$$= \frac{\frac{1}{2} \cos \frac{x}{2} - \frac{1}{2} \cos\left(N + \frac{1}{2}\right)x}{\sin \frac{x}{2}}$$

4. $$\int_\alpha^\beta e^{\overbrace{(a+bi)}^{c}x}\,dx = \int_\alpha^\beta e^{ax}e^{ibx}\,dx =$$

$$= \int_\alpha^\beta e^{ax}\cos bx + i e^{ax}\sin bx\,dx$$

$$= \left(\int_\alpha^\beta e^{ax}\cos bx\,dx\right) + \left(\int_\alpha^\beta e^{ax}\sin bx\,dx\right)i$$

$$= \left.\frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}\right|_\alpha^\beta + \left.\frac{ie^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}\right|_\alpha^\beta$$

$$= \left.\frac{e^{ax}}{a^2+b^2}\left[\underbrace{(a\cos bx + b\sin bx) + i(a\sin x - b\cos bx)}_{(a-bi)(\cos bx + i\sin bx)}\right]\right|_\alpha^\beta$$

$$= \left.\underbrace{\frac{a-bi}{a^2+b^2}}_{\frac{1}{a+bi}}\, e^{ax}e^{ibx}\right|_\alpha^\beta = \left.\frac{1}{c}e^{cx}\right|_\alpha^\beta \quad \checkmark$$

$$\langle e^{inx}, e^{imx} \rangle \quad (n \neq m)$$

$$= \int_{-\pi}^{\pi} e^{inx} \, \overline{e^{imx}} \, dx$$

$$= \overline{\cos mx + i \sin mx}$$

$$= \cos mx - i \sin mx =$$

$$= \cos(-mx) + i \sin(-mx) = e^{-imx}$$

$$= \int_{-\pi}^{\pi} e^{inx} e^{-imx} \, dx$$

$$= \int_{-\pi}^{\pi} \underbrace{e^{inx - imx}}_{e^{i(n-m)x}} \, dx = \frac{1}{i(n-m)} e^{i(n-m)x} \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{i(n-m)} \Big(\underbrace{e^{i(n-m)\pi} - e^{-i(n-m)\pi}}_{= 2i \sin(n-m)\pi = 0} \Big) = 0$$

So e^{inx} and e^{imx} are $\perp$ if

$n, m \in \mathbb{Z}$, $n \neq m$.

(If $n=m$, $\int_{-\pi}^{\pi} e^{inx} \cdot e^{-inx}\, dx = \int_{-\pi}^{\pi} 1\, dx = 2\pi$.)