HWK 3 solutions MA 428

1. $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t \, dt = 0$

$a_n = 0$ for all n because f is <u>odd</u>.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\overbrace{t \sin nt}^{\text{odd}}}_{\text{even}} \, dt = \frac{2}{\pi}\int_0^{\pi} t \sin nt \, dt$$

$$= \frac{2}{\pi}\left[-\frac{t}{n}\cos nt + \frac{1}{n^2}\sin nt\right]_0^{\pi}$$

$$= \frac{2}{\pi} \cdot -\frac{\pi}{n}\underbrace{\cos n\pi}_{(-1)^n} = \underline{\underline{\frac{2}{n}(-1)^{n+1}}}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} t e^{-int} \, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{t\cos nt}_{\text{odd}} - i\underbrace{t \sin nt}_{\text{even}} \, dt$$

$$= \frac{-1}{2\pi} i \int_{-\pi}^{\pi} t \sin nt \, dt = \begin{cases} -\frac{i}{2} b_n & n > 0 \\ \frac{i}{2} b_{|n|} & n < 0 \end{cases}$$

or do this: $c_0 = 0$. If $n \neq 0$,

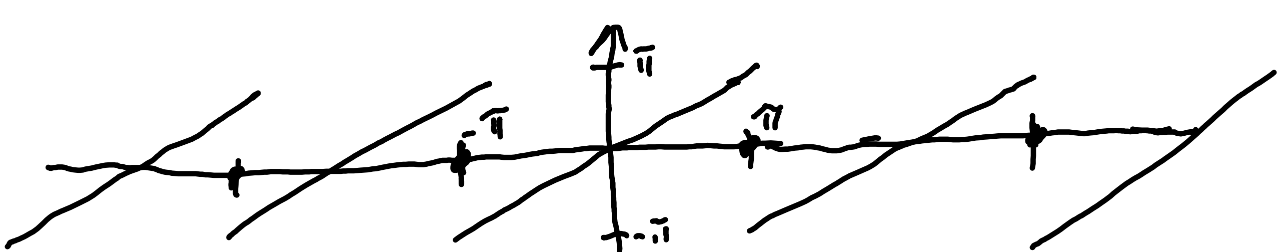
$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{t}_{u}\ \underbrace{e^{-int}\,dt}_{dv} \quad \begin{cases} u = t & dv = e^{-int}dt \\ du = dt & v = \frac{-1}{in}e^{-int} \end{cases}$$

$$= \frac{1}{2\pi}\left[-\frac{t}{in}e^{-int}\right]_{-\pi}^{\pi} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{in}e^{-int}\,dt$$

$$= \frac{1}{2\pi}\left[\frac{-\pi}{in}e^{-in\pi} + \frac{-\pi}{in}e^{in\pi}\right] + \frac{1}{2\pi}\left[\frac{1}{n^2}e^{-int}\right]_{-\pi}^{\pi}$$

$$= \frac{i\pi}{n}\underbrace{\frac{e^{in\pi}+e^{-in\pi}}{2}}_{\cos n\pi} + \frac{1}{2\pi n^2}\Big[\underbrace{e^{-in\pi} - e^{in\pi}}_{-2i\sin n\pi = 0}\Big]$$

$$= \frac{i\pi}{n}\cos n\pi = \frac{i\pi}{n}(-1)^n \qquad n \in \mathbb{Z},\ n \neq 0.$$



$\longrightarrow$ midpoint of jumps $= 0$ at jumps.

2. Plug in PDE: $X''Y + XY'' = 0$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda, \text{ a const.}$$

$$X'' - \lambda X = 0, \quad Y'' + \lambda Y = 0$$

<u>Case $\lambda = 0$:</u> $X(x) = ax + b$, $Y(y) = Ay + B$

Get $u = (ax+b)(Ay+B)$

(generated by xy, x, y, and 1)

<u>Case $\lambda = -k^2 < 0$:</u>

$X(x) = a\cos kx + b\sin kx$ $\quad$ or $ce^{ky} + de^{-ky}$

$Y(y) = A\cosh ky + B\sinh ky$

<u>Case $\lambda = k^2 > 0$:</u> $X(x) = \underbrace{a\cosh kx + b\sinh kx}_{\text{or } ce^{kx} + de^{-kx}}$

$Y(y) = A\cos ky + B\sin ky$

3. $\Delta(r^n \sin n\theta) =$

$$\frac{\partial^2}{\partial r^2}(r^n \sin n\theta) + \frac{1}{r}\frac{\partial}{\partial r}(r^n \sin n\theta) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(r^n \sin n\theta)$$

$$= n(n-1)r^{n-2}\sin n\theta + nr^{n-2}\sin n\theta - n^2 r^{n-2}\sin n\theta = 0$$

Similarly,

$$\Delta(r^n \cos n\theta) = n(n-1)r^{n-2}\cos n\theta + nr^{n-2}\cos n\theta - n^2 r^{n-2}\cos n\theta = 0$$

$$r^n \sin n\theta = r^n \, \frac{e^{in\theta} - e^{-in\theta}}{2i}$$

$$= \frac{1}{2i} \underbrace{r^n e^{in\theta}}_{(re^{i\theta})^n} - \frac{1}{2i} \underbrace{r^n e^{-in\theta}}_{(re^{-i\theta})^n}$$

$$\left\{ \begin{aligned} re^{i\theta} &= \underbrace{r\cos\theta}_{x} + i \underbrace{r\sin\theta}_{y} = x + iy \\ re^{-i\theta} &= r\cos x - i r \sin\theta = x - iy \end{aligned} \right.$$

So $\quad r^n \sin n\theta = \frac{1}{2i}\left[(x+iy)^n - (x-iy)^n\right],$

which expands to be a <u>real</u> polynomial in x and y.

4. a) $u(r,\theta) \equiv 1$ is harmonic and equal to one on the unit circle.

b) $u(r,\theta) = r^1 \sin(1 \cdot \theta) = r \sin\theta$ is harmonic and equal to $\sin\theta$ on the unit circle (when $r=1$).

c) $u(r,\theta) = \frac{1}{2} - \frac{1}{2} r^2 \cos 2\theta$ is harmonic and equal to $\frac{1}{2}(1-\cos 2\theta) = \sin^2\theta$ on the unit circle $(r=1)$.

5. a) $u(x,y) \equiv 1$

b) $u(x,y) = y$

c) $u(x,y) = \frac{1}{2} - \frac{1}{2} \frac{r^2 e^{i2\theta} + r^2 e^{-i2\theta}}{2} =$

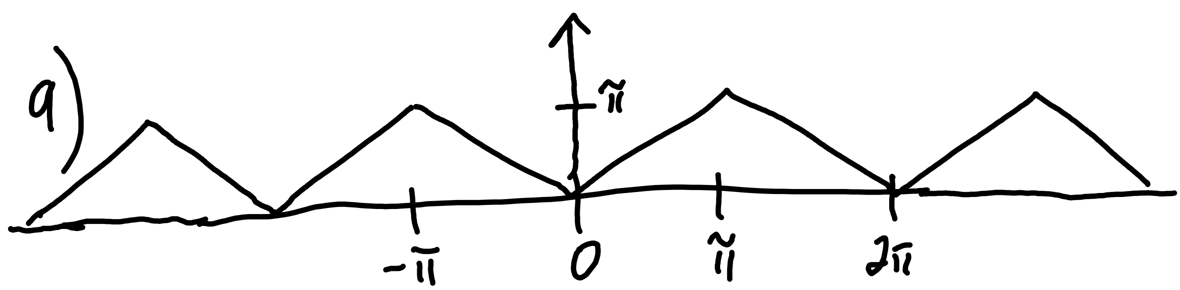
$$= \frac{1}{2} - \frac{1}{2}\,\frac{(re^{i\theta})^2 + (re^{-i\theta})^2}{2}$$

$$= \frac{1}{2} - \frac{1}{4}\left[(x+iy)^2 + (x-iy)^2\right]$$

$$= \frac{1}{2} - \frac{1}{4}\left[2x^2 - 2y^2\right]$$

$$= \frac{1}{2}\left(1 - x^2 + y^2\right)$$

6. a)



b) See calculations for problem 1. These are very similar. Use properties of even and odd functions.

c) $\hat{f}(n) = c_n = \frac{a_n - i b_n}{2}$ when $n > 0$.

So all b_n's $= 0$ (because f is even)

and $a_0 = \frac{\pi}{2}$, $a_n = 2\left[\frac{-1 + (-1)^n}{\pi n^2}\right]$ $n = 1, 2, 3, \ldots$

Taking $\theta = 0$ in the cosine series

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$$

gives

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right].$$

So $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Let $S = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

Notice that $\frac{1}{2^2} S = \frac{1}{(1\cdot 2)^2} + \frac{1}{(2\cdot 2)^2} + \frac{1}{(3\cdot 2)^2} + \cdots$

$$= \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots$$

Hence $\underbrace{\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)}_{\frac{\pi^2}{8}} + \underbrace{\left(\frac{1}{2^2} + \frac{1}{4^2} + \cdots\right)}_{\frac{1}{4}S} = S$

Get $S = \frac{\pi^2}{6}$!

7, Stein p. 63; 15. $\quad w = e^{it}$

Recall that $D_N(t) = \sum_{n=-N}^{N} e^{int} =$

$$(1 + w + \cdots + w^N) + \underbrace{(\bar{\omega} + \bar{\omega}^2 + \cdots + \bar{\omega}^N)}_{\underbrace{\left(\frac{1}{w} + \frac{1}{w^2} + \cdots + \frac{1}{w^N}\right)}_{\frac{1}{w}\left(1 + \frac{1}{w} + \cdots + \frac{1}{w^{N-1}}\right)}} =$$

$$= \frac{1 - w^{N+1}}{1 - w} + \frac{1}{w}\left(\frac{1 - \frac{1}{w^N}}{1 - \frac{1}{w}}\right)$$

$$= \frac{w^{-N} - w^{N+1}}{1 - w}$$ as Stein states.

So $NF_N = D_0 + \cdots + D_{N-1}$

$$= \frac{1}{1-w}\left(\sum_0^{N-1} w^{-n} - w \sum_0^{N-1} w^n\right) =$$

$$= \frac{1}{1-w}\left(\frac{1-\frac{1}{w^N}}{1-\frac{1}{w}} - w\cdot\frac{1-w^N}{1-w}\right)$$

$$= \frac{1}{1-w}\left(\frac{1-w^{-N}}{1-\frac{1}{w}} + \frac{1-w^N}{1-\frac{1}{w}}\right)$$

$$= \frac{2 - w^{-N} - w^N}{2 - w - \frac{1}{w}}$$

$$= \frac{2 - \left(e^{iNt} + e^{-iNt}\right)}{2 - \left(e^{it} + e^{-it}\right)} \qquad \begin{array}{l} \leftarrow 2\cos Nt \\ \leftarrow 2\cos t \end{array}$$

$$= \frac{2\left[1 - \cos Nt\right]}{2\left[1 - \cos t\right]} \qquad \begin{array}{l} \leftarrow \left[2\sin^2 \frac{N}{2}t\right] \\ \leftarrow \left[2\sin \frac{t}{2}\right] \end{array}$$

$$= \frac{\sin^2 \frac{N}{2}t}{\sin^2 \frac{t}{2}} \quad \checkmark$$