MA 428 HWK 5 solutions

1. Separation of variables leads to the Sturm-Liouville problem $X''(x) - \lambda X(x) = 0$ with $X'(0) = 0$ and $X'(L) = 0$, and t-problem $T'(t) - \lambda c^2 T(t) = 0$. The x-problem only has $\neq 0$ solutions when $\lambda = 0$: $X_0(x) = a_0$ and $\lambda = -n^2$, $n = 1, 2, \cdots$; $X_n(x) = \cos \frac{n\pi x}{L}$, $T_0(t) = 1$ and $T_n(t) = e^{-(n\pi/L)^2 c^2 t}$ and we get a sol^n to the heat problem in the form

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-(n\pi/L)^2 c^2 t} \cos \frac{n\pi x}{L}$$

where

the a_n are Fourier cosine series coefficients for $f(x)$ on $[0,L]$ given by $a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{2}$ = ave of f on $[0,L]$ and $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$

$$= \frac{2}{L}\int_0^{L/2} \frac{x}{(L/2)}\cos\frac{n\pi x}{L}\,dx$$

$$+ \frac{2}{L}\int_{L/2}^{L} \frac{L-x}{(L/2)}\cos\frac{n\pi x}{L}\,dx$$

$$= -\frac{8\cos\frac{n\pi}{2}\left(\cos\frac{n\pi}{2}-1\right)}{\pi^2 n^2} = \begin{cases} 0 & n \text{ odd} \\ \frac{-16}{\pi^2 n^2} & n=2,6,10 \\ 0 & n=4,8,12 \end{cases}$$

$$u(x,t) = \frac{1}{2} - \frac{16}{\pi^2}\sum_{n=2,6,10,14,\ldots} \frac{1}{n^2} e^{-\left(\frac{n\pi}{L}\right)^2 c^2 t}\cos\frac{n\pi x}{L}$$

$\lim_{t\to\infty} u(x,t) = \frac{1}{2}$ = ave of f on $[0,L]$.

2. $\lambda=0$: Only get zero solution

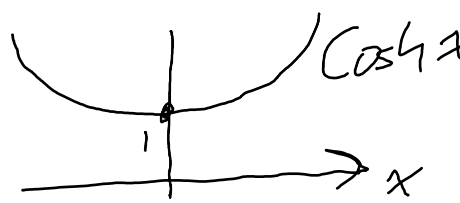
$\underline{\lambda < 0}$: Write $\lambda = -k^2$. $(k>0)$

$$X(x) = A\cosh kx + B\sinh kx$$

$$X'(x) = Ak\sinh kx + Bk\cosh kx$$

$$X'(0) = 0 + Bk \overset{\swarrow \text{want}}{=} 0. \quad \text{So } \boxed{B=0}$$

$$X(\pi) = A\underbrace{\cosh k\pi}_{\neq 0} \overset{\swarrow \text{want}}{=} 0. \quad \text{So } \underline{\underline{A=0}} \text{ too.}$$



Only get the zero sol^n again.

$\left[\cosh x = \frac{e^x + e^{-x}}{2} \text{ is } \underline{\text{never}} \text{ zero}\right]$

$\underline{\lambda > 0}$: Write $\lambda = k^2$. $(k>0)$

$$X(x) = A\cos kx + B\sin kx$$

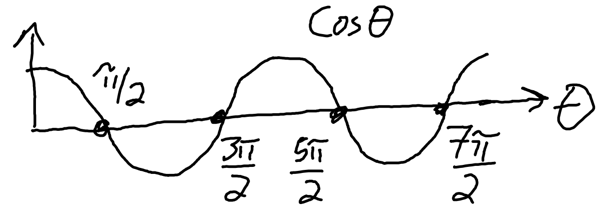
$$X'(x) = -Ak\sin kx + Bk\cos kx$$

$$X'(0) = 0 + Bk\cdot 1 \overset{\swarrow \text{want}}{=} 0. \quad \text{So } \boxed{B=0}$$

$$X(\pi) = A\cos k\pi \overset{\swarrow \text{want}}{=} 0. \quad \text{Don't want } A=0 \text{ too.}$$

So we need $\cos k\pi = 0$,

i.e, $k\pi = \frac{(\text{odd})\pi}{2}$



$k = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots \qquad \lambda = k^2 = \frac{(2n-1)^2}{2^2} \qquad n = 1, 2, 3, \cdots$

For $\lambda = \frac{(2n-1)^2}{4}$: $\underline{X}(x) = \cos \frac{(2n-1)}{2} x$.

3. $(1-z)(1+z+z^2+\cdots+z^N)$

$$
\begin{array}{rl}
= & 1 + z + z^2 + \cdots + z^N \\
- & \quad\;\; z + z^2 + \cdots + z^N + z^{N+1} \\
\hline
 & 1 - z^{N+1}
\end{array}
$$

So $1 + z + \cdots + z^N = \dfrac{1 - z^{N+1}}{1 - z}$

Diff'tiate: $0 + 1 + 2z + 3z^2 + \cdots + Nz^{N-1}$

$$= \frac{-(N+1)z^N(1-z) - (-1)(1-z^{N+1})}{(1-z)^2}$$

$$= \frac{1 + N z^{N+1} - (N+1)z^N}{(1-z)^2}$$

Multiply by z:

$$z + 2z^2 + 3z^3 + \cdots + Nz^N = \frac{z + Nz^{N+2} - (N+1)z^{N+1}}{(1-z)^2}$$

Finally, let $z = e^{i\theta}$, use $(e^{i\theta})^n = e^{in\theta}$

$$= \cos n\theta + i \sin n\theta$$

and take the <u>real</u> part to get

$$\cos\theta + 2\cos 2\theta + \cdots + N\cos N\theta =$$

$$\mathrm{Re}\left[\underbrace{\frac{e^{i\theta} + N e^{i(N+2)\theta} - (N+1)e^{i(N+1)\theta}}{(1-e^{i\theta})^2}}_{S'}\right]$$

$$S' = S' \cdot \frac{\overbrace{(e^{-i\theta/2}/2i)^2}^{=\frac{e^{-i\theta}}{-4}}}{(e^{-i\theta/2}/2i)^2} = \frac{1 + Ne^{i(N+1)\theta} - (N+1)e^{iN\theta}}{-4\sin^2\theta/2}$$

$$\text{Ans.} = \mathrm{Re}\, S' = \frac{1 + N\cos(N+1)\theta - (N+1)\cos N\theta}{-4\sin^2\theta/2}$$

4. $D_N(x) = \sum_{n=-N}^{N} e^{inx} = \underset{\uparrow}{1} + \sum_{n \neq 0} \underset{\uparrow}{e^{inx}}$

$2\pi = \int_{-\pi}^{\pi} 1\,dx \qquad \int_{-\pi}^{\pi} = 0$

So $\int_{-\pi}^{\pi} D_N(x)\,dx = 2\pi$ is easy.

We also know that $D_N(x) = \dfrac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$

Stein's hint: $\dfrac{1}{\sin\frac{x}{2}} - \dfrac{1}{\left(\frac{x}{2}\right)} = \dfrac{\frac{x}{2} - \sin\frac{x}{2}}{\frac{x}{2}\sin\frac{x}{2}} \xrightarrow[x \to 0]{L'H \#1}$

$$\lim_{x \to 0} \frac{\frac{1}{2} - \frac{1}{2}\cos\frac{x}{2}}{\frac{1}{2}\sin\frac{x}{2} + \frac{x}{2}\cdot\frac{1}{2}\cos\frac{x}{2}} \overset{L'H \#2}{=}$$

$$\lim_{x \to 0} \frac{\frac{1}{4}\sin\frac{x}{2}}{\frac{1}{4}\cos\frac{x}{2} + \frac{1}{4}\cos\frac{x}{2} - \frac{x}{8}\sin\frac{x}{2}} = \frac{0}{\left(\frac{1}{2}\right)} = 0$$

So $\quad f(x) = \left(\frac{1}{(\sin \frac{x}{2})} - \frac{1}{(\frac{x}{2})} \right)$ and

$$f(x) \cdot \sin (N + \tfrac{1}{2}) x$$

are continuous on $[-\pi, \pi]$ if we give them a value of zero at $x = 0$.

Note that $\sin(N+\frac{1}{2})x = \cos Nx \sin\frac{x}{2} + \sin Nx \cos\frac{x}{2}$

So $\displaystyle\int_{-\pi}^{\pi} \sin (N+\tfrac{1}{2})x \; f(x)\, dx$

$$= \underbrace{\int_{-\pi}^{\pi} \cos Nx \left(f(x) \sin\frac{x}{2} \right) dx}_{\substack{\pi \cdot \text{ Fourier cosine coeff} \\ \text{for continuous fcn} \\ f(x) \sin\frac{x}{2}}} + \underbrace{\int_{-\pi}^{\pi} \sin Nx \left(f(x) \cos\frac{x}{2} \right) dx}_{\substack{\pi \cdot \text{ Fourier sine coeff} \\ \text{for continuous} \\ \text{fcn } f(x) \cos\frac{x}{2}}}$$

and Bessel's ineq $\Longrightarrow$ these go to zero as

N goes to ∞. Now we have

$$\underbrace{\int_{-\pi}^{\pi} \underbrace{\frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}}_{D_N(x)}}_{2\pi} - \frac{\sin(N+\frac{1}{2})}{\frac{x}{2}}\,dx \xrightarrow{N\to\infty} 0$$

so $$\lim_{N\to\infty} \int_{-\pi}^{\pi} \frac{\sin(N+\frac{1}{2})x}{\frac{x}{2}}\,dx = 2\pi$$

But $$\int_{-\pi}^{\pi} \underbrace{\frac{\sin(N+\frac{1}{2})x}{\frac{x}{2}}}_{\frac{\text{odd}}{\text{odd}} = \text{even}}\,dx = 2\int_{0}^{\pi} \frac{\sin(N+\frac{1}{2})x}{\frac{x}{2}}\,dx$$

and letting $\theta = (N+\frac{1}{2})x$ $\quad d\theta = (N+\frac{1}{2})dx$

$x = \frac{\theta}{N+1/2}$

$$\begin{cases} \text{When } x=0,\ \theta=0 \\ \text{When } x=\pi,\ \theta=(N+\frac{1}{2})\pi \end{cases}$$

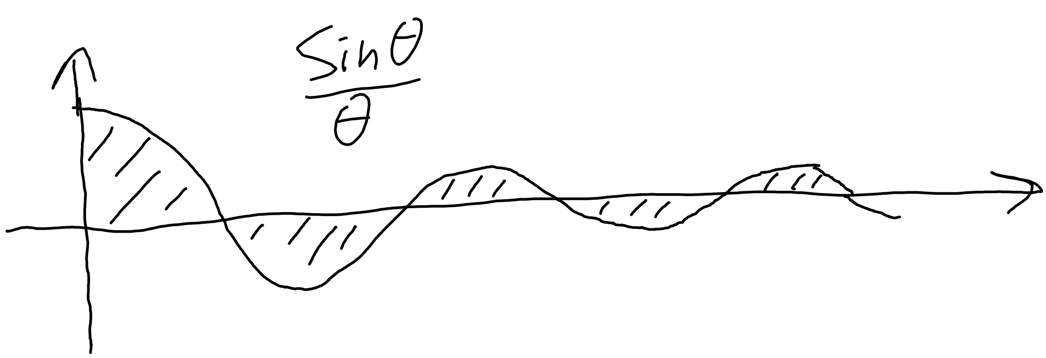
transforms the integral

to $$4\int_0^{(N+\frac{1}{2})\pi} \frac{\sin\theta}{\frac{\theta}{(N+\frac{1}{2})}} \frac{d\theta}{N+\frac{1}{2}}$$

$$= 4\int_0^{(N+\frac{1}{2})\pi} \frac{\sin\theta}{\theta}\, d\theta$$

Letting $N \to \infty$ yields that

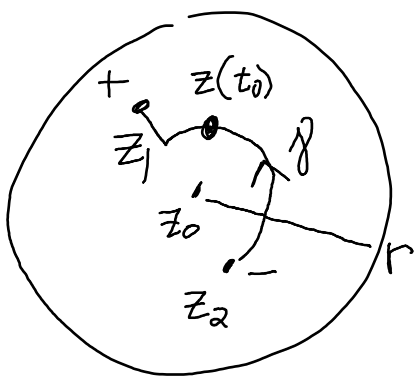
$$4\int_0^{\infty} \frac{\sin\theta}{\theta}\, d\theta = 2\pi \quad \checkmark$$



<u>Remark</u>: Alternating series test shows that $\lim\limits_{B\to\infty} \int_0^B \frac{\sin\theta}{\theta}\, d\theta$ exists.

5. If u is nonvanishing on $D_r(z_0)-\{z_0\}$, suppose there are points z_1 and z_2 in $D_r(z_0)-\{z_0\}$ where $u(z_1)>0$ and $u(z_2)<0$.

γ misses z_0



Connect z_1 and z_2 by a continuous curve γ parametrized by $z(t)=x(t)+iy(t)$, $a\le t\le b$. Now the Intermediate value theorem applied to the continuous fcn $u(x(t),y(t))$ on $[a,b]$ implies that there is a $t_0\in(a,b)$ where this fcn is zero. $\lightning$ So u is either always positive or always negative on $D_r(z_0)-\{z_0\}$. Harmonic fcns satisfy the averaging property. If a harmonic fcn

u had an isolated zero at z_0 (meaning that there is a radius $r>0$ such that z_0 is the <u>only</u> zero of u in $D_r(z_0)$), then the averaging property on a disc of radius ρ with $0<\rho<r$ about z_0 gives

$$0 = u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{u(z_0+\rho e^{it})}_{\substack{\text{always } >0 \\ \text{or} \\ \text{always } <0}}\, dt$$

The integral <u>cannot</u> be zero. This contradiction implies that u cannot have an isolated zero at z_0.