MA 428 HWK 6 solutions

1. $$\begin{cases} \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(t)^2\,dt = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \\ \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} F(t)^2\,dt = 2A_0^2 + \sum_{n=1}^{\infty}(A_n^2 + B_n^2) \end{cases}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\big(f(t)+F(t)\big)^2\,dt = 2(a_0+A_0)^2 + \sum_{1}^{\infty}\left[(a_n+A_n)^2+(b_n+B_n)^2\right]$$

$$-\quad \frac{1}{\pi}\int_{-\pi}^{\pi}\big(f(t)-F(t)\big)^2\,dt = 2(a_0-A_0)^2 + \sum_{1}^{\infty}\left[(a_n-A_n)^2+(b_n-B_n)^2\right]$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} 4f(t)F(t)\,dt = 2\cdot 4a_0A_0 + \sum_{1}^{\infty}\left(4a_nA_n + 4b_nB_n\right) \checkmark$$

<u>Note</u>: We used the important fact that $f \longmapsto$ (Fourier coeff of f) is a <u>linear</u> map.

2. $$\frac{1}{2\pi}\int_{-\pi}^{\pi}(f+\lambda g)\overline{(f+\lambda g)}\,dt = \sum_{-\infty}^{\infty}(\hat{f}(n)+\lambda\hat{g}(n))\overline{(\hat{f}(n)+\lambda\hat{g}(n))}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(f-\lambda g)\overline{(f-\lambda g)}\,dt = \sum_{-\infty}^{\infty}(\hat{f}(n)-\lambda\hat{g}(n))\overline{(\hat{f}(n)-\lambda\hat{g}(n))}$$

$-$

Using $|z+\lambda w|^2 = (z+\lambda w)\overline{(z+\lambda w)}$

$$= \underbrace{z\bar{z}}_{|z|^2} + \lambda w\bar{z} + \bar{\lambda}\bar{w}z + \underbrace{\lambda\bar{\lambda}}_{|\lambda|^2=1}\underbrace{w\bar{w}}_{|w|^2}$$

$$= |z|^2 + 2\,\mathrm{Re}\,\bar{\lambda}z\bar{w} + |w|^2$$

and $|z-\lambda w|^2 = |z|^2 - 2\,\mathrm{Re}\,\bar{\lambda}z\bar{w} + |w|^2.$

Note: $\overline{\bar{\lambda}z\bar{w}} = \lambda w\bar{z}$

and $\zeta + \bar{\zeta} = 2\,\mathrm{Re}\,\zeta$

Get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 4\,\mathrm{Re}\,\bar{\lambda} f(t)\overline{g(t)}\,dt$$

$$= \sum_{-\infty}^{\infty} 4\,\mathrm{Re}\,\bar{\lambda}\hat{f}(n)\overline{\hat{g}(n)}$$

Since $\int u+iv\,dt = \int u\,dt + i\int v\,dt$, we know $\mathrm{Re}\int u+iv\,dt = \int \mathrm{Re}(u+iv)\,dt$.

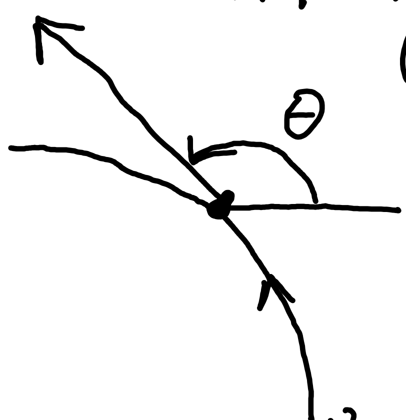
You showed earlier in the semester that $\lambda\int(u+iv)\,dt = \int\lambda(u+iv)\,dt$. So

$$\mathrm{Re}\,\bar{\lambda}\underbrace{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f\bar{g}\,dt - \sum_{-\infty}^{\infty}\hat{f}(n)\bar{\hat{g}}(n)\right)}_{re^{i\theta}} = 0$$

Take $\lambda = e^{i\theta}$ to see $\underbrace{(\overline{e^{i\theta}})}_{e^{-i\theta}} re^{i\theta} = r = 0$ ✓

3. Let the origin of your coordinate system be the point where the rover starts.

$$\vec{v} = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath}$$



t = time

$\gamma : (x(t), y(t))$

In time Δt, approximate

$$\Delta x \approx \dot{x}\,\Delta t = (\cos\theta)\,|\vec{v}|\,\Delta t$$

$$\Delta y \approx \dot{y}\,\Delta t = (\sin\theta)\,|\vec{v}|\,\Delta t$$

Get $x \approx \sum \Delta x$. Use the trapezoid rule to approximate

$\dot{y}(t) \approx |\vec{v}| \sin\theta$

$$\int_0^T x\,dy = \int_0^T x(t)\,\dot{y}(t)\,dt.$$

Green's theorem shows that

$$\int_\gamma x\,dy = \int_\gamma \underset{=0}{P}\,dx + \underset{=x}{Q}\,dy$$

$$= \iint_\Omega \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{1-0} dx\,dy = \text{Area}(\Omega)$$

4. The complex Fourier coeff of

$$f(t) = \frac{\pi}{\sin \pi\alpha} e^{i(\pi - x)\alpha} \quad \text{are}$$

$$\hat{f}(n) = \frac{1}{2\pi}\frac{\pi}{\sin \pi\alpha}\int_0^{2\pi} \underbrace{e^{i(\pi-x)\alpha} e^{-inx}}_{e^{i\pi\alpha}\cdot e^{-i(n+\alpha)x}} dx$$

$$= \frac{e^{i\pi\alpha}}{2\sin \pi\alpha}\int_0^{2\pi} e^{-i(n+\alpha)x}\,dx =$$

$$= \frac{e^{i\pi\alpha}}{2\sin\pi\alpha} \cdot \frac{1}{-i(n+\alpha)} \Big[e^{-i(n+\alpha)x} \Big]_0^{2\pi}$$

$$= \frac{e^{i\pi\alpha}}{2\sin\pi\alpha} \cdot \frac{1}{-i(n+\alpha)} \Big(\underbrace{e^{-i(n+\alpha)2\pi}}_{\underbrace{e^{-in2\pi}}_{=1} \cdot e^{-i\alpha 2\pi}} - 1 \Big)$$

$$= \frac{1}{(n+\alpha)\sin\pi\alpha} \underbrace{\frac{e^{-i\alpha\pi} - e^{i\alpha\pi}}{-2i}}_{=\sin\pi\alpha}$$

$$= \frac{1}{n+\alpha} \quad \checkmark$$

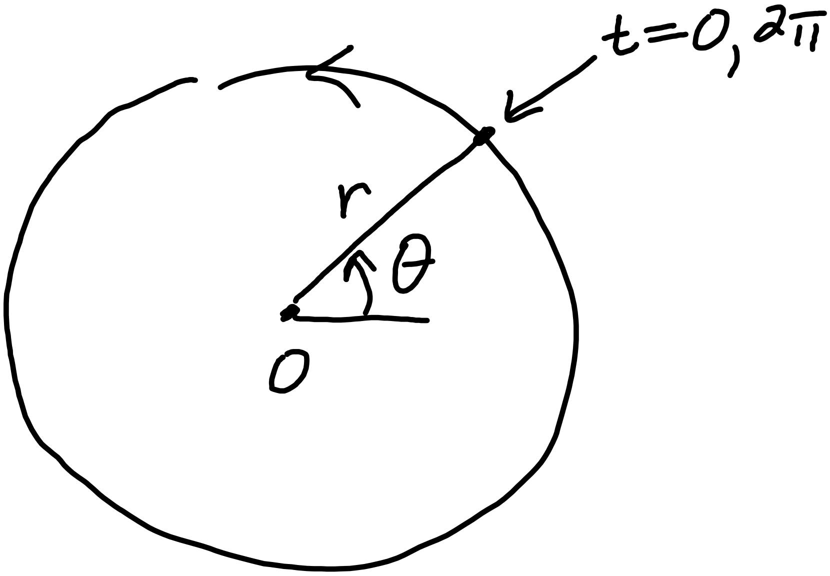
Parseval's: $\displaystyle \frac{1}{2\pi}\int_0^{2\pi} \frac{\pi^2}{\sin^2\pi\alpha} \underbrace{\left| e^{i(\pi-x)\alpha} \right|^2}_{=1} dx = \frac{\pi^2}{\sin^2\pi\alpha} = \sum_{-\infty}^{\infty} \frac{1}{(n+\alpha)^2}$

$(\alpha \notin \mathbb{Z})$

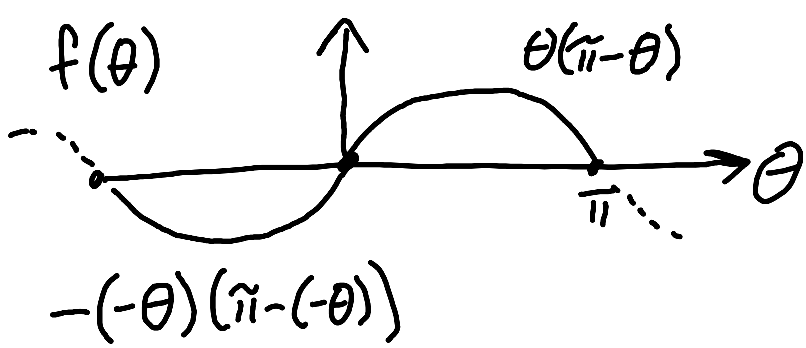
5. $$x(t)+iy(t) = (\alpha\cos t - \beta\sin t) + i(\beta\cos t + \alpha\sin t)$$

$$= \underbrace{(\alpha+i\beta)}_{re^{i\theta}}\underbrace{(\cos t + i\sin t)}_{e^{it}}$$

$$= re^{i(\theta+t)}$$



6.



odd
2π-periodic

f odd, so all cosine coeff $=0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{f(\theta)}_{\text{odd}}\ \underbrace{\sin n\theta}_{\text{odd}}}_{\text{even}}\, d\theta = \frac{2}{\pi}\int_0^{\pi} \theta(\pi-\theta)\sin n\theta\, d\theta$$

= MAPLE: (2/Pi)*int(t*(Pi-t)*sin(n*t),t=0..Pi);

$$= \begin{cases} \frac{8}{\pi}\cdot\frac{1}{n^3} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Parseval's: $\left(\frac{8}{\pi}\right)^2 \sum_{\substack{n \text{ odd} \\ n\geq 1}} \frac{1}{n^6} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)^2\, d\theta$

$$= \frac{2}{\pi}\int_0^{\pi} \theta^2(\pi-\theta)^2\, d\theta.$$ Get $\sum_{\text{odd}} \frac{1}{n^6}$.

Use $\sum_{n \text{ even}} \frac{1}{n^6} = \frac{1}{2^6}\sum_1^{\infty} \frac{1}{n^6}$ and

$\sum_{n \text{ odd}} \frac{1}{n^6} + \sum_{n \text{ even}} \frac{1}{n^6} = \sum_1^{\infty} \frac{1}{n^6}$ to get $\sum_1^{\infty} \frac{1}{n^6}$.