MA 428 HWK 7 solutions

1. Let $\varepsilon > 0$.

Since $\displaystyle\int_{-\infty}^{\infty} |f(x)|\,dx = \int_{-N}^{N} |f(x)|\,dx + \int_{|x|>N} |f(x)|\,dx$

and $\displaystyle\lim_{N\to\infty} \int_{-N}^{N} = \int_{-\infty}^{\infty}$, it follows that

$\displaystyle\int_{|x|>N} |f(x)|\,dx \to 0$ as $N \to \infty$. So

$\exists N$ such that $\displaystyle\int_{|x|>N} |f(x)|\,dx < \frac{\varepsilon}{2}$.

Notice that $\displaystyle\left|\int_{|x|>N} f(x) \sin wx\,dx\right|$

$$\le \int_{|x|>N} |f(x)|\,\underbrace{|\sin wx|}_{\le 1}\,dx \le \int_{|x|>N} |f(x)|\,dx < \frac{\varepsilon}{2}$$

too.

Now $|\alpha(w)| =$

$$\left| \int_{-N}^{N} \underbrace{f(x)}_{u} \sin wx \, dx + \int_{|x|>N} f(x) \sin wx \, dx \right|$$

$$\leq \underbrace{\left| \int_{-N}^{N} \underbrace{f(x)}_{u} \underbrace{\sin wx \, dx}_{dv} \right|} + \frac{\varepsilon}{2}$$

$$= \left| f(x) \left[-\frac{1}{w} \cos wx \right] \Bigg|_{-N}^{N} + \int_{-N}^{N} \frac{1}{w} \cos wx \cdot f'(x) dx \right|$$

$$\leq \frac{1}{|w|} \left[|f(N)| + |f(-N)| + \int_{-N}^{N} |f'(x)| dx \right]$$

and this last term goes to zero as $|w| \to \infty$, and hence can be made less than $\frac{\varepsilon}{2}$. So $\alpha(w) \to 0$ as $|w| \to \infty$.

2. $$\int_{-\infty}^{\infty} \underbrace{e^{-x^2}}_{\text{even}} dx = \int_{-1}^{1} e^{-x^2} dx + \underbrace{2\int_{1}^{\infty} e^{-x^2} dx}_{\leq 2\int_{1}^{\infty} e^{-x} dx}$$

$$\underbrace{\leq 2\int_{1}^{\infty} e^{-x} dx}_{= 2\lim_{B \to \infty} \left[-e^{-x}\right]_1^B = 2e^{-1}}$$

So $$\int_{-\infty}^{\infty} e^{-x^2} dx \leq \underbrace{\int_{-1}^{1} \underbrace{e^{-x^2}}_{\leq 1} dx}_{\leq 2 \cdot 1} + \frac{2}{e} \leq 2 + \frac{2}{e}$$

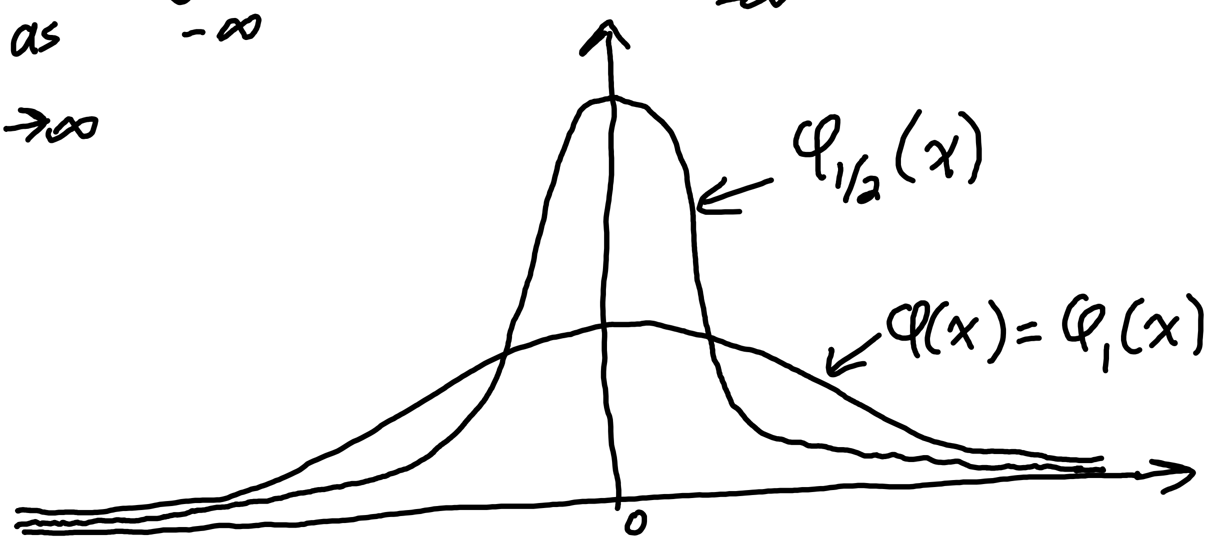
3. $\displaystyle\int_{-B}^{B} \frac{1}{t}\varphi(\underbrace{x/t}_{u})\,dx$ $\qquad du = \frac{1}{t}\,dx \quad (t>0)$

when $x=-B$, $u=\frac{-B}{t}$

$x=B$, $u=B/t$

$$= \int_{-B/t}^{B/t} \varphi(u)\,du$$

$$\xrightarrow[\text{as } B\to\infty]{} \int_{-\infty}^{\infty} \varphi(u)\,du = \frac{1}{c}\int_{-\infty}^{\infty} e^{-x^2}dx = \frac{c}{c} = 1.$$



(Area under both $= 1$.)

If $\delta>0$,

$$\int_{\delta}^{\infty} \varphi_t(x)\,dt = \int_{\delta}^{\infty} \frac{1}{t}\varphi(x/t)\,dt = \int_{\delta/t}^{\infty} \varphi(u)\,du \longrightarrow 0$$

as $t \to 0$ (because $B = \delta/t \to \infty$ and

$$\lim_{B\to\infty} \int_B^\infty e^{-x^2}\,dx = 0 \Big).$$

4. Notice that $f(0) = f(0)\underbrace{\int_{-\infty}^{\infty} \varphi_t(x)\,dx}_{=1}$, so

$$\left| f(0) - \int_{-\infty}^{\infty} f(x)\,\varphi_t(x)\,dx \right| =$$

$$\left| \int_{-\infty}^{\infty} \big(f(0) - f(x)\big)\,\varphi_t(x)\,dx \right|$$

$$\le \underbrace{\int_{-\delta}^{\delta} |f(0) - f(x)|\,\varphi_t(x)\,dx}_{I_1}$$

$$+ \underbrace{\int_{|x|>\delta} |f(0) - f(x)|\,\varphi_t(x)\,dx}_{I_2}$$

Since f is bounded, there is an $M>0$ such that $|f(x)|<M$ for all x.

Let $\varepsilon>0$. Since f is continuous at $x=0$, there is a $\delta>0$ such that $|f(0)-f(x)|<\frac{\varepsilon}{2}$ when $|x|<\delta$. Now

$$|I_1| \leq \int_{-\delta}^{\delta} \underbrace{|f(0)-f(x)|}_{<\frac{\varepsilon}{2}} \varphi_t(x)\,dx$$

$$\leq \frac{\varepsilon}{2} \underbrace{\int_{-\delta}^{\delta} \varphi_t(x)\,dx}_{<\int_{-\infty}^{\infty}\varphi_t(x)\,dx = 1} < \frac{\varepsilon}{2}.$$

$$|I_2| \leq \int_{|x|>\delta} \underbrace{\big(|f(0)|+|f(x)|\big)}_{<2M} \varphi_t(x)\,dx \leq$$

$$\leq 2M \int_{|x|>\delta} \underset{\uparrow \text{ even}}{\varphi_t(x)}\,dx = 2 \cdot 2M \int_{\delta}^{\infty} \varphi_t(x)\,dx$$

$\longrightarrow 0$ as $t \to 0$ from prob. 3.

So $|I_2|$ can be made $< \frac{\varepsilon}{2}$ too by taking t sufficiently small.

5. Suppose f is a continuous 2π-periodic function. We want to prove the Lemma:

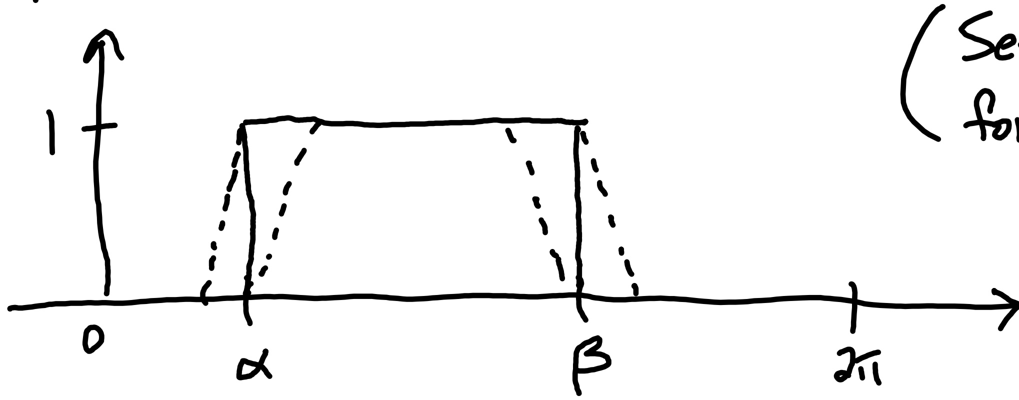
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\theta_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

following Weyl's ideas. Note that the formula is true if $f(\theta) \equiv 1$. If $f(\theta) = e^{im\theta}$ for $m \in \mathbb{Z}$, then we are given that the left side is zero, and the right side is $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{im\theta}\,d\theta = 0$ too.

So the the lemma is true for Fourier basis functions, and hence for trig polys $\sum_{-N}^{N} A_n e^{in\theta}$. Now we may procede exactly as in the proof of Weyl's equidistribution theorem to use Féjer's theorem to get the lemma for continuous functions, and finally approximate the 2π-periodic extension of the characteristic function χ for the interval $[\alpha, \beta]$ via

2π-periodic extensions of the "trapezoidal functions" above and below

(See Lesson 23 for details)



to obtain

$$\frac{1}{N}\left\{\# \, e^{i\theta_n} \in (\alpha,\beta) \text{ arc on } C_1(0),\ 1\le n\le N\right\}$$

$$\longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} \chi \, d\theta = \frac{\beta-\alpha}{2\pi}$$

and this exactly what it means for $e^{i\theta_n}$ to be equidistributed.

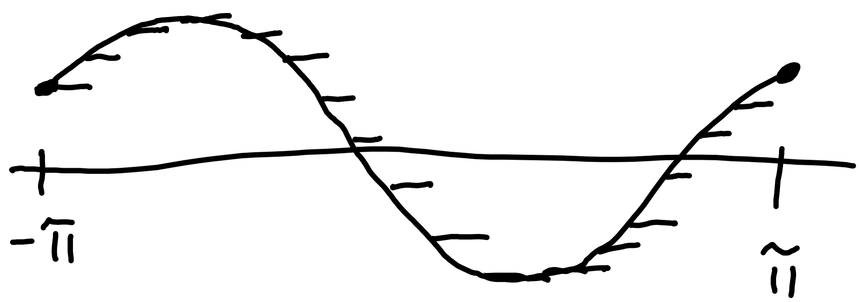
To prove the converse, note that $e^{i\theta_n}$ equidistributed is equivalent to

$$\frac{1}{N}\sum_{n=1}^{N} f(e^{i\theta_n}) \to \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta \qquad (*)$$

for $f(\theta)=$ the 2π-periodic extension of the characteristic function $\chi_{(\alpha,\beta)}$ for the interval (α,β). Hence, if $e^{i\theta_n}$ are equidistributed, then $(*)$ is true for step fcns $\sigma(\theta)=\sum_{i=1}^{M} c_i \chi_{(\alpha_i,\beta_i)}(\theta)$.

Given a continuous 2π-periodic fcn f and $\varepsilon>0$, the <u>uniform</u> continuity of f allows us to find a step fcn

$\sigma(\theta)$ such that $|f(\theta) - \sigma(\theta)| < \frac{\varepsilon}{3}$ for all θ:



Now $\left| \frac{1}{N} \sum_{n=1}^{N} f(e^{i\theta_n}) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta \right|$

$\leq \left| \frac{1}{N} \sum_{n=1}^{N} \left(f(e^{i\theta_n}) - \sigma(e^{i\theta_n}) \right) \right| +$

$\left| \frac{1}{N} \sum_{n=1}^{N} \sigma(e^{i\theta_n}) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \sigma(\theta)\, d\theta \right|$

$+ \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sigma(\theta) - f(\theta) \right) d\theta \right|$

and the first term is $< \frac{\varepsilon}{3}$ because

it is $\leq \displaystyle \frac{1}{N}\sum_{n=1}^{N} \underbrace{|f(e^{i\theta_n}) - \sigma(e^{i\theta_n})|}_{< \frac{\varepsilon}{3}}$

$< \frac{1}{N}\cdot\left(N\cdot\frac{\varepsilon}{3}\right)$ ✓, the second

term can be made less than $\frac{\varepsilon}{3}$

by taking N sufficiently large

because (*) is true for step

functions; and the third term is

$< \frac{\varepsilon}{3}$ because it is $\leq$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{|f(\theta) - \sigma(\theta)|}_{< \frac{\varepsilon}{3}}\, d\theta \leq \frac{1}{2\pi}\cdot\left(2\pi\cdot\frac{\varepsilon}{3}\right)$$ ✓.

Hence $(*)$ is true for continuous fcns f. Finally, Weyl's criterion follows because it is the statement of $(*)$ using $f(\theta) = e^{im\theta}$ for $m = 1, 2, 3, \ldots$