

Lecture 24 Solutions to Midterm.

1. $x^7, \cos 3x, 1 \perp$ on $[-\pi, \pi]$

$$\int_{-\pi}^{\pi} \underbrace{x^7}_{\substack{\uparrow \\ \text{odd}}} \underbrace{\cos 3x}_{\substack{\uparrow \\ \text{even}}} dx = 0$$

odd

$$\int_{-\pi}^{\pi} \underbrace{x^7 \cdot 1}_{\text{odd}} dx = 0$$

$$\int_{-\pi}^{\pi} \underbrace{1 \cdot \cos 3x}_{\text{even}} dx = 2 \int_0^{\pi} \cos 3x dx$$

$$= 2 \left[\frac{1}{3} \sin 3x \right]_0^{\pi} = 0$$

$$\text{If } x^7 f(x) = \left[c_0 + c_1 x^7 + c_2 \cos 3x \right] x^7$$

$$\int_{-\pi}^{\pi} x^7 f(x) dx = c_1 \int_{-\pi}^{\pi} (x^7)^2 dx$$

$$= c_1 \left[\frac{1}{15} x^{15} \right]_{-\pi}^{\pi}$$

$$c_1 = \frac{\int_{-\pi}^{\pi} x^7 f(x) dx}{\left(\frac{2}{15} \pi^{15} \right)}$$

2. $\underline{X}'' + \lambda \underline{X} = 0$ on $[0, \pi]$

$$\underline{X}'(0) = 0, \quad \underline{X}(\pi) = 0$$

$\lambda > 0$; $\lambda = k^2$

$$r^2 + k^2 = 0$$

$$r = \pm ki$$

$$\underline{X}(x) = \underline{c_1 \cos kx} + \cancel{c_2 \sin kx}$$

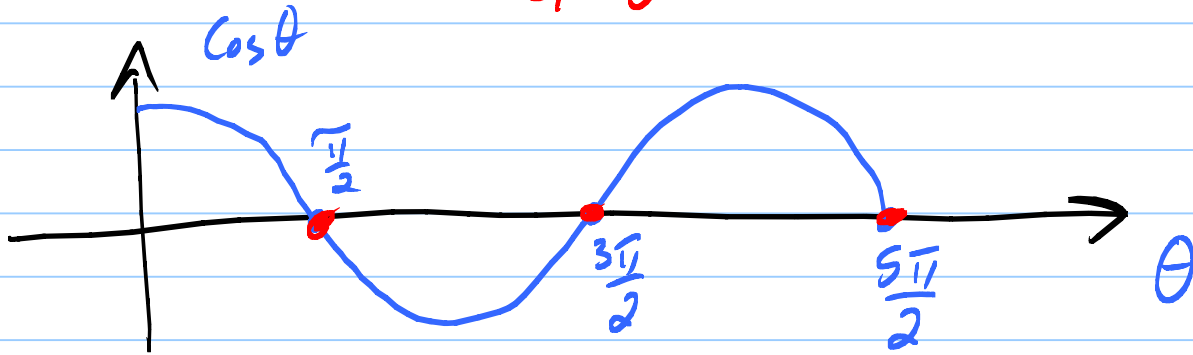
$$\underline{X}'(x) = -c_1 k \sin kx + \cancel{c_2 k \cos kx}$$

Need: $\underline{X}'(0) = 0 + c_2 k \cdot 1 \stackrel{\text{want}}{=} 0$

$$\boxed{c_2 = 0}$$

and $X(\tilde{\eta}) = c_1 \underbrace{\cos k\tilde{\eta}}_{\substack{\text{need this} = 0. \\ \leftarrow \text{want } 0}} = 0$

$\uparrow$
don't want $c_1 = 0$



Need $k\tilde{\eta} = (\text{odd}) \frac{\pi}{2} \stackrel{\text{or}}{=} (n + \frac{1}{2})\pi$
 $n = 0, 1, 2, \dots$

So $\lambda = k^2 = \left[\left(n + \frac{1}{2} \right) \right]^2$, $n = 0, 1, 2, 3, \dots$

For n : $X_n = \underbrace{\cos \left(n + \frac{1}{2} \right) x}_{\substack{\text{eigenfunctions} \\ \text{for } \frac{d^2}{dx^2}}}$

$X'' = -\lambda X$

Orthogonal and complete! Parseval's Identity.

$$3. (1-z)(1+z+z^2+\dots+z^N) =$$

$$= \frac{1+z+\dots+z^N}{z+z^2+\dots+z^N+z^{N+1}} = \frac{1-z^{N+1}}{1-z}$$

$$1+z+\dots+z^N = \frac{1-z^{N+1}}{1-z}$$

$$\cos \theta = \operatorname{Re} e^{i\theta}$$

$$\cos n\theta = \operatorname{Re} e^{in\theta}$$

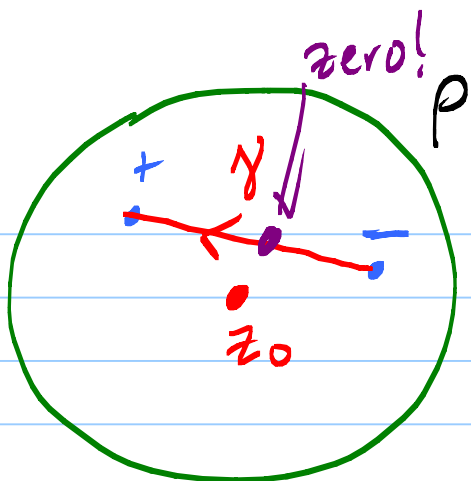
$$\text{So } 1 + \cos \theta + \dots + \cos N\theta$$

$$= \operatorname{Re} [1 + e^{i\theta} + e^{i2\theta} + \dots + e^{iN\theta}]$$

$$= \operatorname{Re} [1 + (e^{i\theta}) + (e^{i\theta})^2 + \dots + (e^{i\theta})^N]$$

$$= \operatorname{Re} \left[\frac{1 - e^{i(N+1)\theta}}{1 - e^{i\theta}} \right]$$

4.



z_0 isolated
zero of u

Shrink ρ so it is

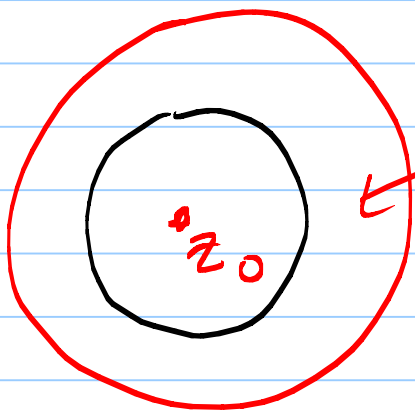
the only zero on $D_\rho(z_0)$.

Can make γ miss z_0 . $\gamma: \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

$0 \leq t \leq 1$. $\underbrace{u(x(t), y(t))}_{h(t)}$ continuous.

$h(0) < 0$, $h(1) > 0$. I.V.T. gives

t_0 , $0 < t_0 < 1$ where $h(t_0) = 0$. ⚡



always + on $D_\rho(z_0) - \{z_0\}$.

[or take $-u$ in
place of u]

$$0 = u(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \underbrace{u(z_0 + r e^{it})}_{> 0} dt > 0$$