Math 428

Practice problems

1. Assume that $g(x)$ is a real valued C^1-smooth function on the whole real line such that $\int_{-\infty}^{\infty} |g(x)|\ dx$ is finite. Show that

$$\alpha(w) = \int_{-\infty}^{\infty} g(x) \sin wx\ dx$$

goes to zero as $w \to \infty$. Hint: Do integration by parts on a finite piece of the integral after showing that the part of the integral outside $[-N, N]$ goes to zero as $N \to \infty$.

2. Define a function $f(x)$ to be used in problems 2-4 as follows:

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x \leq 1 \\ 0 & 1 < x \end{cases}$$

Compute the complex Fourier transform

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\ dx$$

and the Fourier sine transform

$$[\mathcal{F}_s f](w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\ \sin wx\ dx.$$

3. Solve the heat problem $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ on $[0, \infty)$ for $x \geq 0$ and $t \geq 0$, with $u(x, 0) = f(x)$ for $x \geq 0$, and $u(0, t) = 0$ for all $t \geq 0$.

4. Solve the heat problem $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ on $(-\infty, \infty)$ for $-\infty \leq x \leq \infty$ and $t \geq 0$, with $u(x, 0) = f(x)$ for all x.

5. Prove Weyl's Criterion, that given a sequence of angles θ_n, the points $e^{i\theta_n}$ are equidistributed on the unit circle if and only if

$$\frac{1}{N} \sum_{n=1}^{N} e^{im\theta_n}$$

tends to zero as N tends to infinitey for each positive integer m.

Hints: The limit for a postive integer m implies the limit for negative integer $-m$ because one is the conjugate of the other. The $m = 0$ case is special (and different). Follow the proof of Weyl's Theorem: true for Fourier basis functions implies true for trigonometric polynomials implies true for continuous 2π-periodic functions implies ...