MA 428 Exam 2 solutions

1. As in Lesson 30, take the Fourier sine transform of $u(x,t)$ in the space variable:

$$\hat{u}(s,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \sin(sx)\, dx$$

Then

$$\frac{\partial \hat{u}}{\partial t}(s,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \underbrace{\frac{\partial u}{\partial t}(x,t)}_{= \frac{\partial^2 u}{\partial x^2}(x,t)} \sin(sx)\, dx$$

$$= -s^2 \hat{u}(s,t) + \sqrt{\frac{2}{\pi}}\, \underbrace{u(0,t)}_{=0}\, s = -s^2 \hat{u}(s,t).$$

Hence $\hat{u}(s,t) = C(s) e^{-s^2 t}$ where $C(s)$ is an unknown function of s. But

$$\hat{u}(s,0) = C(s) \cdot 1 = \sqrt{\frac{2}{\pi}} \int_0^\infty \underbrace{f(x)}_{=u(x,0)} \sin(sx)\, dx$$

So

$$C(s) = \sqrt{\frac{2}{\pi}} \left(\int_1^2 (x-1) \sin(sx)\, dx + \int_2^3 (3-x) \sin(sx)\, dx \right)$$

= via integration by parts...

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{2\sin(2s) - \sin s - \sin(3s)}{s^2}$$

Now we "undo" the Fourier sine transf by applying it again:

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \underbrace{c(s)e^{-s^2 t}}_{\hat{u}(s,t)} \sin(xs)\,ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{2\sin(2s) - \sin s - \sin(3s)}{s^2} e^{-s^2 t} \sin(sx)\,ds$$

Interesting to note how smooth the fcn inside the integral is as $s \to 0$!

2. If f is continuous and supported in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\hat{f}(s) \equiv 0$ when $|s| > N$, then the Fourier coefficients for f on $[-\pi, \pi]$,

$$C_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$$

$$= 2 \underbrace{\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-inx} \, dx \right)}_{\hat{f}(n) = 0} = 0$$

if $|n| > N$.

Fourier transform! (not C_n coeff)

This means that the Fourier series for f on $[-\pi, \pi]$ is the finite sum $\sum_{n=-N}^{N} c_n e^{inx}$, i.e., a trig poly of degree N (or less).

<u>Note</u>: We do <u>not</u> know that the Fourier series of a merely continuous function f converges to f. (In fact, Stein constructs an example where it doesn't. See pp. 83-87.)

However, if the Fourier coefficients are sumable in the sense that $\sum_{n=-\infty}^{\infty} |c_n| < \infty$, which they certainly are in case the sum is <u>finite</u>, then Corollary 2.3 on page 41 yields that the Fourier series <u>converges uniformly to f</u>, so $f(x) = \sum_{-N}^{N} c_n e^{inx}$ is a trig poly!

<u>Claim</u>: A trig poly that vanishes on an open subinterval $(a,b) \subset [-\pi, \pi]$ must be identically zero.

Indeed, $P_N(x) = \sum_{-N}^{N} c_n e^{inx} =$

$$\sum_{-N}^{N} (a_n + i b_n)(\cos nx + i \sin nx)$$

$$= \left(\sum_{-N}^{N} \alpha_n \cos nx + \beta_n \sin nx \right) + i \left(\sum_{-N}^{N} A_n \cos nx + B_n \sin nx \right)$$

$= u(x) + i\,v(x)$ where u and v

are linear combinations of sines and cosines, which can be expanded in a power series about a point $x_0 \in (a,b)$ that has an infinite radius of convergence. In particular, the Taylor series for u and v about $x = x_0$ converge uniformly on $[-\pi, \pi]$. The series for $u + iv$ converges uniformly to $P_N = f$ there.

But, if $u \equiv 0$ and $v \equiv 0$ near x_0, Taylor's formula shows that all the Taylor coeff for u and v about x_0 are zero! We have shown that $f \equiv 0$ on $[-\pi, \pi]$.

We assumed that f vanishes outside $[-\pi, \pi]$, so $f \equiv 0$ on $\mathbb{R}$.

Similarly, if f is supported in $(-B, B)$, we may choose $L > 0$ such that $(-B, B) \subset [-\frac{L}{2}, \frac{L}{2}]$ and repeat the argument above using Fourier series $f(x) \sim \sum_{-\infty}^{\infty} c_n e^{in\pi x/L}$

where $c_n = \frac{1}{L} \int_{-L}^{L} f(x) e^{-in\pi x/L} \, dx$

to see that $\hat{f}$ compactly supported $\Longrightarrow f \equiv 0$.

3. D'Albert's method yields that

$$u(x,t) = \varphi(x+ct) + \psi(x-ct).$$

Compute $\frac{\partial u}{\partial t} = c\varphi'(x+ct) - c\psi(x-ct).$

We need

$$\begin{cases} u(x,0) = \varphi(x) + \psi(x) \overset{\text{want}}{=} e^{-x^2} & (A) \\ \frac{\partial u}{\partial t}(x,0) = \underbrace{c\varphi'(x) - c\psi'(x) \overset{\text{want}}{=} 0}_{\varphi'(x) = \psi'(x) \quad (B')} & (B) \end{cases}$$

B' shows that φ and ψ differ by constant, so $\varphi(x) = \psi(x) + C$ for some C, and (A) becomes

$$2[\varphi(x) + C] = e^{-x^2}.$$

Taking $c=0$ yields a solution to (A), (B):

$$\varphi(x) = \frac{1}{2} e^{-x^2}$$

$$\psi(x) = \frac{1}{2} e^{-x^2}$$

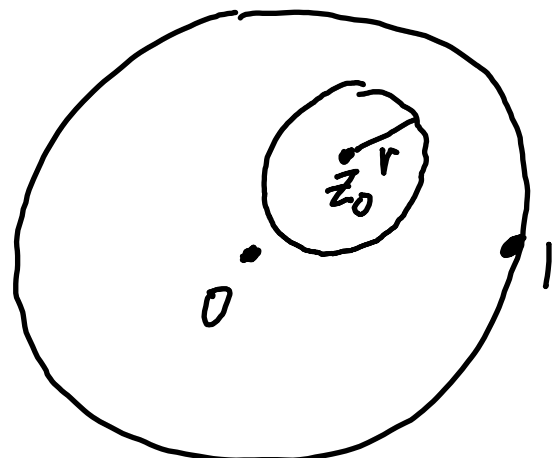
and

$$u(x,t) = \frac{1}{2}\left[e^{-(x+ct)^2} + e^{-(x-ct)^2} \right]$$

solves the wave equation with the prescribed initial conditions.

4. Let $U(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} P(r,\theta,t)\, u(e^{it})\, dt$

be the Poisson integral of the boundary values of u. We know that U is continuous on the closed unit disc and harmonic inside and $u \equiv \bar{U}$ on the unit circle. Furthermore, being harmonic, $\bar{U}$ satisfies the averaging property on circles:



and we may let $r \nearrow (1-|z_0|)$ and use the <u>uniform</u> continuity of $\bar{U}$ on $\overline{D_1(0)}$ to conclude that $\bar{U}$ also satisfies the averaging property on the circles pictured in the problem.

We now follow the ideas of Lesson 16.

Since $u-\tilde{U}$ is continuous on the closed and bounded $\overline{D_1(0)}$, it assumes its maximum value M at some point $z_0 \in \overline{D_1(0)}$. We know that $u-\tilde{U} \equiv 0$ on the unit circle. Suppose $M>0$ is assumed at z_0 inside $D_1(0)$. Then $u(z_0)-\tilde{U}(z_0) = M$ is the average of a continuous fcn on the circle pictured that is $\leq M$ on the circle. This forces us to conclude that $u-\tilde{U} \equiv M$ on the circle. But

$u(e^{it}) - U(e^{it}) = 0$ at the point of tangency. Hence M must be zero, and this contradiction yields that $u - U$ cannot have a positive max in $D_1(0)$. So $u - U \leq 0$ on $\overline{D_1(0)}$. Repeat this argument using $U - u$ in place of $u - U$ to also obtain that $U - u \leq 0$ on $\overline{D_1(0)}$. We conclude that

$u \equiv U$, and consequently,

u is harmonic on $D_1(0)$.