Math 428 Exam

Each problem is worth 25 points.

1. Explain why

$$\int_{-\pi}^{\pi} e^{-x^2} \sin 3x \; dx = 0$$

without trying to find an antiderivative (which is impossible).

2. Find all possible *positive* values of λ that allow non-zero solutions to the boundary value problem

$$X''(x) + \lambda X(x) = 0$$

on $[0, \pi]$ with $X(0) = 0$ and $X'(\pi) = 0$. For each such λ, find a non-zero solution.

3. Given that

$$1 + z + z^2 + \cdots + z^N = \frac{1 - z^{N+1}}{1 - z}$$

when z is a complex number unequal to one, find a short closed expression for

$$1 - \sin\theta + \sin 2\theta - \sin 3\theta + \cdots + (-1)^N \sin N\theta$$

that does not involve a sum of length N. (Do not try to simplify the expression you get.)

Hints: What happens if you replace z by $-z$? Note that $\sin n\theta$ is the imaginary part of $e^{in\theta}$.

4. Assume that

$$S(x) = \sum_{n=1}^{\infty} B_n \sin nx$$

is the sum of the Fourier sine series on $[0, \pi]$ for the function $f(x)$ that is equal to one on the interval $[0, \pi/2)$ and zero on $[\pi/2, \pi]$.

a) Write an integral formula for B_n and evaluate the integral (but don't simplify it or try to see a pattern).

b) Graph the function that the Fourier sine series converges to on $[-3\pi, 3\pi]$, being careful at jumps.

c) Evaluate the sum

$$\sum_{n=1}^{\infty} {B_n}^2.$$

Hint: The Fourier sine series is equal to the full Fourier series of the function you graphed in part (b).

Formula sheet, OVER

Formulas

Fourier sine series for $f(x)$ on $[0, \pi]$ is $\sum_{n=1}^{\infty} B_n \sin nx$ where

$$B_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \ dx$$

Full Fourier series for $f(x)$ on $[-\pi, \pi]$ is $a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \ dx$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \ dx$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \ dx$$

The complex version is $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \ dx$$

Facts: $c_0 = a_0$, and if $n \geq 1$: $c_n = \frac{a_n - ib_n}{2}$ and $c_{-n} = \frac{a_n + ib_n}{2}$

Bessel's inequalities

$$2a_0^2 + \sum_{n=1}^{\infty} ({a_n}^2 + {b_n}^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$