MA 428 Exam 1 solutions

1. $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{\pi} 1\,dx = \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\sin x\right]_0^{\pi} = 0 - 0 = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} \sin nx\,dx$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} = \frac{1}{\pi n}\left[-\cos n\pi - (-1)\right]$$

$$= \frac{1}{\pi n}\left[1 - (-1)^n\right] = \begin{cases} \frac{2}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Parseval's: $$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\frac{1}{\pi}\int_0^{\pi} 1^2\,dx = 2\left(\frac{1}{2}\right)^2 + \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \left(\frac{2}{\pi n}\right)^2$$

$$\underline{1} = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

2. $P_0(x) \equiv A$ is a trig poly of degree zero: $A = Ae^{i0\cdot x}$. Among all trig polys of degree zero, the Fourier partial sum $S_0(x) = a_0$ is the best approximation to $g(x)$ in $L^2[-\pi, \pi]$. So the

answer is $A = a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx$

= the average value of g on $[-\pi, \pi]$.

3. Write $\lambda = k^2$. The characteristic eqn is

$r^2 + k^2 = 0$. So $r = \pm ki$ and the gen^{e}

sol^{n} is $X(x) = c_1 \cos kx + c_2 \sin kx$.

$$X'(x) = -c_1 k \sin kx + c_2 k \cos kx$$

Want $X'(0) = -c_1 \cdot 0 + c_2 k \cdot 1 = c_2 k \overset{\text{want}}{=} 0$

So $\boxed{c_2 = 0}$

and $X'(\pi) = -c_1 k \sin k\pi \overset{\text{want}}{=} 0$

To get a non-zero sol^{n}, we need

$c_1 \neq 0$, so we must have $\sin k\pi = 0$.

$\sin\theta$ is zero for positive θ only when $\theta = n\pi$, $n=1,2,3,\dots$. Hence, we need $K\pi = n\pi$, $n=1,2,3,\dots$, i.e.,

$\boxed{K=n}$ $n=1,2,3,\dots$

<u>Ans</u>: $\lambda_n = K^2 = n^2$, $n=1,2,3,\dots$

$$X_n(x) = c\cos nx \quad (c\neq 0)$$

4. $1-\cos\theta + \cos 2\theta + \cdots + (-1)^N \cos N\theta$

$$= \mathrm{Re}\left[1 - e^{i\theta} + e^{i2\theta} + \cdots + (-1)^N e^{iN\theta}\right]$$

$$= \mathrm{Re}\left[1 + (-z) + (-z)^2 + \cdots + (-z)^N\right] =$$

where $z = e^{i\theta}$

$$= \mathrm{Re}\left[\frac{1-(-z)^{N+1}}{1-(-z)}\right]$$

$$= \mathrm{Re}\left[\frac{1-(-1)^{N+1}e^{i(N+1)\theta}}{1+e^{i\theta}}\right]$$

<u>Path 1:</u>

$$= \mathrm{Re}\left[\frac{1-(-1)^{N+1}e^{i(N+1)\theta}}{1+e^{i\theta}} \cdot \underbrace{\frac{e^{-i\theta/2}/2}{e^{-i\theta/2}/2}}_{1}\right]$$

$$= \mathrm{Re}\left[\frac{\frac{1}{2}\left(e^{-i\theta/2}-(-1)^{N+1}e^{i(N+\frac{1}{2})\theta}\right)}{\cos\frac{\theta}{2}}\right]$$

$$= \frac{\cos\frac{\theta}{2} - (-1)^{N+1}\cos\left(N+\frac{1}{2}\right)\theta}{2\cos\frac{\theta}{2}}$$

Path 2 :

$$= \mathrm{Re}\left[\frac{\left(1-(-1)^{N+1}\cos(N+1)\theta\right) + i(-1)^{N+1}\sin(N+1)\theta}{(1+\cos\theta)+i\sin\theta}\right]$$

$$= \mathrm{Re}\left[\frac{A+Bi}{C+Di}\cdot\frac{C-Di}{C-Di}\right]$$

$$= \frac{AC+BD}{C^2+D^2}$$ where $C = 1+\cos\theta$, $D = \sin\theta$

$A = 1-(-1)^{N+1}\cos(N+1)\theta$ and $B = (-1)^{N+1}\sin(N+1)\theta$.