MA 428 Exam 2 solutions

1. $$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_0^1 \underset{\uparrow \atop u}{x} \underbrace{e^{-isx}\, dx}_{dv} + \int_1^2 \underbrace{(2-x)}_{u} \underbrace{e^{-isx}\, dx}_{dv} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left[x \cdot \left(\frac{-1}{is} e^{-isx} \right) \Big|_0^1 - \int_0^1 \left(-\frac{1}{is} e^{-isx} \right) dx \right.$$

$$\left. + (2-x)\left(\frac{-1}{is} e^{-isx} \right) \Big|_1^2 - \int_1^2 \left(-\frac{1}{is} e^{-isx} \right)(-dx) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{-e^{-is}}{is} - \left[\frac{1}{(is)^2} e^{-isx} \right]_0^1 \right.$$

$$\left. - \left(\frac{-e^{-is}}{is} \right) + \left[\frac{1}{(is)^2} e^{-isx} \right]_1^2 \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{s^2} \left(e^{-is} - 1 \right) - \frac{1}{s^2} \left(e^{-i2s} - e^{-is} \right) \right] =$$

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \left(\frac{-e^{-i2s} + 2e^{-is} - 1}{s^2} \right)$$

Since $\sqrt{2\pi}\,\hat{f}(s) = \int_{-\infty}^{\infty} = \int_0^{\infty} f(x)\, \underset{\substack{\uparrow \\ \cos(sx) - i\sin(sx)}}{e^{-isx}}\, dx$,

we see that $\tilde{\mathcal{F}}_c f = \sqrt{\frac{2}{\pi}}\sqrt{2\pi}\, \mathrm{Re}\, \hat{f}(s)$.

So $\left[\tilde{\mathcal{F}}_c f\right](s) = \sqrt{\frac{2}{\pi}}\, \mathrm{Re}\left(\frac{-e^{-i2s} + 2e^{-is} - 1}{s^2} \right)$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{-\cos 2s + 2\cos s - 1}{s^2} \right)$$

2. Let $\hat{u}(s,t) = \tilde{\mathcal{F}}_c\left[u(x,t)\right](s)$

$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} u(x,t) \cos(sx)\, dx$. Then

$$\frac{\partial \hat{u}}{\partial t} = \sqrt{\frac{2}{\pi}} \int_0^\infty \underbrace{\frac{\partial u}{\partial t}(x,t)}_{= \frac{\partial^2 u}{\partial x^2}} \cos(sx)\, dx$$

$$= \tilde{\mathcal{F}}_c\left[\frac{\partial^2 u}{\partial x^2}\right] = -s^2 \underbrace{\tilde{\mathcal{F}}_c[u]}_{\hat{u}} - \sqrt{\frac{2}{\pi}} \underbrace{\frac{\partial u}{\partial x}(0,t)}_{=0}$$

So $\boxed{\frac{\partial \hat{u}}{\partial t} = -s^2 \hat{u}}$ and $\hat{u}(s,t) = C(s) e^{-s^2 t}$.

Plug in $t=0$ to see $C(s) = \hat{u}(s,0) = \left[\tilde{\mathcal{F}}_c f\right](s) =$

$\sqrt{\frac{2}{\pi}}\left(\frac{-\cos 2s + 2\cos s - 1}{s^2}\right)$. Finally,

$$u(x,t) = \tilde{\mathcal{F}}_c \hat{u} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{-\cos 2s + 2\cos s - 1}{s^2}\right) e^{-s^2 t} \cos(sx)\, ds$$

3. Let $\hat{u}(s,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-isx}\, dx$.

$$\frac{\partial \hat{u}}{dt} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\frac{\partial u}{\partial t}(x,t)}_{= \frac{\partial^2 u}{\partial x^2}} e^{-isx}\, dx$$

$$= \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = -s^2 \mathcal{F}[u]$$

So $\frac{\partial \hat{u}}{\partial t} = -s^2 \hat{u}$ and $\hat{u}(s,t) = C(s) e^{-s^2 t}$

where, this time, $C(s)$ needs to be an arbitrary complex valued function of s.

Plug in $t=0$ to see $C(s) = \hat{u}(s,0) =$

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{u(x,0)}_{f(x)} e^{-isx}\, dx = \hat{f}(s)$. Finally,

$$u(x,t) = \mathcal{F}^{-1}\left[\hat{f}(s)e^{-s^2t}\right]$$

$$\underline{\underline{\text{ans}}} \rightarrow = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\left(\frac{-e^{-i2s}+2e^{-is}-1}{s^2}\right)e^{-s^2t}\cdot e^{isx}\,ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\underbrace{\frac{-\cos 2s+2\cos s-1}{s^2}}_{\text{even}} + i\,\underbrace{\frac{\sin 2s - 2\sin s}{s^2}}_{\text{odd}}\right]\underset{\substack{\uparrow \\ \text{even}}}{e^{-s^2t}}\cdot$$

$$\cdot\left[\underbrace{\cos sx}_{\text{even}} + i\,\underbrace{\sin sx}_{\text{odd}}\right]ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{-\cos 2s+2\cos s-1}{s^2}\cos(sx) - \frac{\sin 2s-2\sin s}{s^2}\sin(sx)\right]$$

$$\cdot e^{-s^2t}\,ds$$

4. D'Alembert's method calls for making the change of variables $w = x+ct$, $v = x-ct$ to transform the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ to the easy equation $\frac{\partial^2 u}{\partial w \partial v} = 0$, which has solution $u(w,v) = \varphi(w) + \psi(v)$, yielding

$$u(x,t) = \varphi(x+ct) + \psi(x-ct).$$

Note that $\frac{\partial u}{\partial t}(x,t) = c\varphi'(x+ct) - c\psi'(x-ct)$, so, to satisfy the initial conditions

$$\begin{cases} u(x,0) = \varphi(x) + \psi(x) \overset{\text{want}}{=} e^{-x^2} & \text{(A)} \\ \frac{\partial u}{\partial t}(x,0) = c\varphi'(x) - c\psi'(x) = 0 & \text{(B)} \end{cases}$$

we need eqns (A) and (B) to hold.

(B) shows that $\varphi' \equiv \psi'$, so

$$\psi(x) = \varphi(x) + K, \text{ where } K = \text{const.}$$

Now (A) yields $\varphi(x) + [\varphi(x) + K] = e^{-x^2}$,

which we can solve by setting $K=0$ and $\varphi(x) = \frac{1}{2} e^{-x^2}$. Note, since $K=0$, we have $\psi(x) = \varphi(x) = \frac{1}{2} e^{-x^2}$ too.

Finally, $u(x,t) = \frac{1}{2}\left[e^{-(x+ct)^2} + e^{-(x-ct)^2} \right]$

which represents the hump e^{-x^2} breaking into two half size humps traveling in $\pm$ directions at speed c.

<u>Remarks</u> $\hat{f}(s)$ and $\tilde{\mathcal{F}}_c[f]$ are not singular at $s=0$ because, if you write out

$$e^{-i2s} = 1 + (-i2s) + \frac{(-i2s)^2}{2!} + \cdots$$

$$e^{-is} = 1 + (-is) + \frac{(-is)^2}{2!} + \cdots$$

and collect terms in

$$-e^{-i2s} + 2e^{-is} - 1 = 0 + 0\cdot s + 1\cdot s^2 + \cdots$$

$$= s^2\cdot(\text{convergent power series on } \mathbb{R}),$$

you can deduce that $\hat{f}(s)$ and $\tilde{\mathcal{F}}_c f$ are C^∞-smooth on $\mathbb{R}$ when given the proper values at $s=0$.