Math 428 Final exam

1. Let $f(x) = \begin{cases} 1 & -\pi < x < 0 \\ 3 & 0 < x < \pi. \end{cases}$, and let $a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ be the Fourier series for f on $[-\pi, \pi]$. Evaluate

$$\sum_{n=0}^{\infty} (-1)^n a_n = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi.$$

2. If the Fourier sine Series of $f(x) = x^4$ on $[0, \pi]$ is $\sum_{n=1}^{\infty} b_n \sin nx$, evaluate the sum of the infinite series $\sum_{n=1}^{\infty} b_n^2$.

3. Suppose $f(x) = 1 + \cos x + \sum_{n=1}^{\infty} \frac{1}{n^2} \sin nx$ on $[-\pi, \pi]$.

Explain why f is continuous.

Evaluate $\int_{-\pi}^{\pi} f(x) \cos 5x \, dx$ and $\int_{-\pi}^{\pi} f(x) \sin 5x \, dx$.

4. Compute the Fourier transform of the even function

$$f(x) = \begin{cases} 1 & -3 < x < 3 \\ 0 & \text{elsewhere.} \end{cases}$$

What is the Fourier transform of the Fourier transform of f?

Be careful to give the value of the function for *every* real number.

5. Suppose that $u(r, \theta)$ is the harmonic function with continuous boundary values on the unit circle given in polar coordinates by

$$u(1, \theta) = \sin \theta + \cos 2\theta.$$

Evaluate $u\left(\frac{1}{2}, \frac{\pi}{2}\right)$.

6. Suppose that f is continuous on $[0, \pi]$ and that $\int_0^{\pi} f(x) \sin nx \, dx = 0$ for every positive integer n. Explain why f must be identically zero.

7. Let $g(x) = \sin 3\pi x$. If $u(x,t)$ is the solution to the problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= 4\frac{\partial^2 u}{\partial x^2} && 0 < x < 2,\ t > 0 \\ u(x,0) &= g(x) && 0 < x < 2 \\ \frac{\partial u}{\partial t}(x,0) &= 0 && 0 < x < 2 \\ u(0,t) = u(2,t) &= 0 && 0 \le t, \end{aligned}$$

then $u(\frac{1}{2}, \frac{1}{12}) =$

8. Let $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin \frac{(2n+1)\pi x}{2}$ and $u(x,t)$ be the solution to

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} && 0 \le x \le 2,\ t > 0 \\ u(0,t) = u(2,t) &= 0 && t > 0 \\ u(x,0) &= f(x) && 0 \le x \le 2 \end{aligned}$$

Then $u(x,t) =$

A. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} e^{-n^2\pi^2 t/4} \sin \frac{(2n+1)\pi x}{2}$

B. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \cos \frac{n^2\pi^2 t}{4} \sin \frac{(2n+1)\pi x}{2}$

C. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} e^{-(2n+1)^2\pi^2 t/4} \sin \frac{(2n+1)\pi x}{2}$

D. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \cos \frac{(2n^2+1)^2\pi^2 t}{4} \sin \frac{(2n+1)\pi x}{2}$

E. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \cos \frac{(2n+1)\pi t}{2} \sin \frac{(2n+1)\pi x}{2}$