Math 428 Final exam

Each problem is 25 points

Formula sheet on back

1. Let $f(x) = \begin{cases} 1 & -\pi \le x < 0 \\ x & 0 \le x \le \pi. \end{cases}$ and let $s(x)$ denote the function on $[-\pi, \pi]$ to which the Fourier series for f converges. Express $s(x)$ as a piecewise defined function on $[-\pi, \pi]$, being careful at jumps and $\pm\pi$.

2. Suppose that $a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ is the Fourier cosine series of a continuous function $f(x)$ on $[0, \pi]$. Find a formula that expresses the sum $2a_0^2 + \sum_{n=1}^{\infty} a_n^2$ in terms of an integral involving f over the interval $[0, \pi]$.

3. Suppose that f is continuous on $[0, \pi]$ and that $\int_0^\pi f(x) \cos nx \, dx = 0$ for every positive integer n. Explain why f must be a constant function on $[0, \pi]$.

4. Explain why the Fourier transform of the Fourier transform of a Schwartz function $f(x)$ is $f(-x)$.

5. It is not hard to show that xe^{-x^2} is a Schwartz function. (You may assume this.) Let $f(x) = \displaystyle\int_{-\infty}^{\infty} te^{-t^2} \sin(tx) \, dt$. Find the Fourier transform of f.
Hint: Explain why $\int_{-\infty}^{\infty} te^{-t^2} \cos(tx) \, dt$ is equal to zero without attempting to do any difficult computations. Realize f as the Fourier transform of something and consider the effect of applying the Fourier transform twice.

6. Use the fact that $e^{i\alpha} e^{i\beta} = e^{i(\alpha+\beta)}$ to deduce a trig formula for $\sin(\alpha + \beta)$.

7. Explain why

$$\int_{-\pi}^{\pi} f(t) \sin[(N + 1/2)t] \, dt$$

tends to zero as $N \to \infty$ if f is continuous on $[-\pi, \pi]$. ($N \in \mathbb{Z}^+$)

8. Assuming the result of Problem 7, explain why

$$\int_{-\pi}^{\pi} \frac{\sin[(N + 1/2)t]}{\sin t/2} \left(F(x + t) - F(x)\right) \, dt$$

tends to zero as $N \to \infty$ if F is continuously differentiable and 2π-periodic on $\mathbb{R}$. (Here, x is a fixed point in $[-\pi, \pi]$ and $N \in \mathbb{Z}^+$.)

Formulas

Fourier sine series for $f(x)$ on $[0,\pi]$ is $\sum_{n=1}^{\infty} B_n \sin nx$ and the cosine series is $A_0 + \sum_{n=1}^{\infty} A_n \cos nx$ where $A_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$, $A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$, and $B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

Full Fourier series for $f(x)$ on $[-\pi,\pi]$ is $a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ where

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}$$

The complex version is $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$

Facts: $c_0 = a_0$, and if $n \geq 1$: $c_n = \frac{a_n - ib_n}{2}$ and $c_{-n} = \frac{a_n + ib_n}{2}$

Parseval's identities (Bessel's inequality with $\leq$ in place of $=$)

$$2a_0^2 + \sum_{n=1}^{\infty}({a_n}^2 + {b_n}^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx \quad \text{and} \quad \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$$

Fourier transform of f is $(\mathcal{F}f)(s) = \hat{f}(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-isx}\,dx$

Inverse Fourier transform of g is $(\mathcal{F}^{-1}g)(x) = \check{g}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(s)e^{isx}\,ds$

Fourier cosine transform of f is $(\mathcal{F}_c f)(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(sx)\,dx$

Fourier sine transform of f is $(\mathcal{F}_s f)(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(sx)\,dx$

The Plancherel formula is $\int_{-\infty}^{\infty} |f(x)|^2\,dx = 2\pi\int_{-\infty}^{\infty} |\hat{f}(s)|^2\,ds$