MA 428 Midterm Exam solutions

1. $$\underbrace{\int_{-\pi}^{\pi}}_{\substack{\text{symmetric} \\ \text{about } 0}} \underbrace{\underbrace{e^{-x^2}}_{\text{even}} \underbrace{\sin 3x}_{\text{odd}}}_{\text{odd}} \, dx = 0$$

2. Let $\lambda = k^2$. $\quad X'' + k^2 X = 0$

Char eqn: $r^2 + k^2 = 0 \quad \boxed{r = \pm k i}$

$$X(x) = A \sin kx + B \cos kx$$

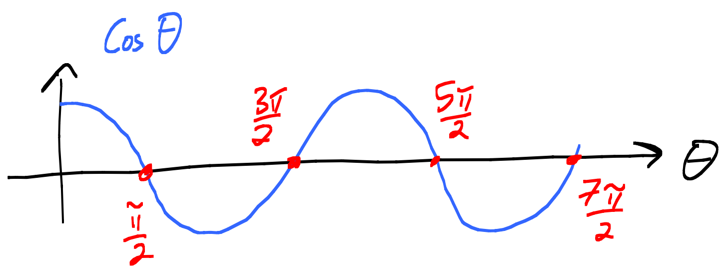
$$X'(x) = Ak \cos kx - Bk \sin kx$$

Need:

$$\begin{cases} X(0) = B \overset{\text{want}}{=} 0 \\ X'(\pi) = Ak \cos k\pi \overset{\text{want}}{=} 0 \end{cases}$$

$B = 0$. Don't want $A = 0$ too.

So need $\boxed{\cos k\pi = 0}$



Need

$k\pi = \frac{\pi}{2} + n\pi, \quad n = 0, 1, 2, \ldots$

$k = (n + \frac{1}{2}), \quad n = 0, 1, 2,$

Ans: $\lambda = k^2 = \underline{\underline{(n + \frac{1}{2})^2}}, \quad n = 0, 1, 2, \ldots$

$X_n(x) = \sin kx = \underline{\underline{\sin (n + \frac{1}{2}) x}}$ ← taking $A = 1$.

3. $$\underbrace{1 - (e^{i\theta}) + (e^{i\theta})^2 - (e^{i\theta})^3 + \cdots + (-1)^N (e^{i\theta})^N}_{} = \frac{1 - (-e^{i\theta})^{N+1}}{1 - (-e^{i\theta})}$$

$$S = 1 - e^{i\theta} + e^{i2\theta} - e^{i3\theta} + \cdots + (-1) e^{iN\theta} \quad //$$

$$1 - \sin\theta + \sin 2\theta - \sin 3\theta + \cdots + (-1)^N \sin N\theta$$

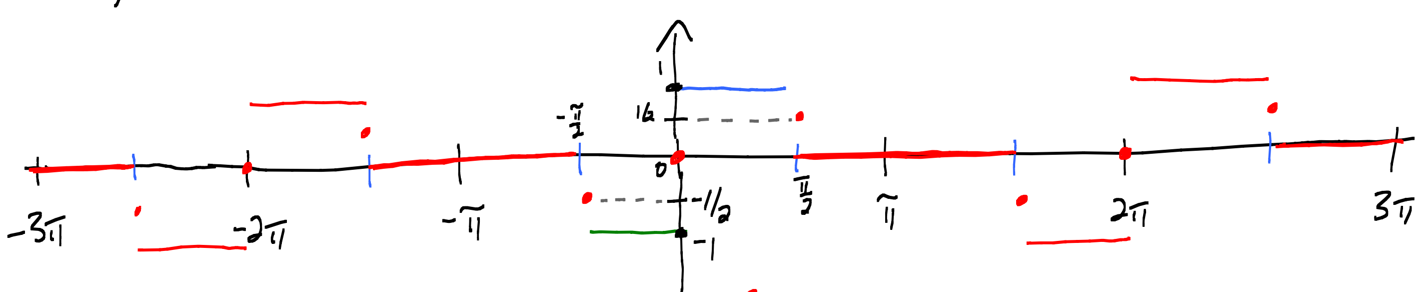
$$= 1 + \operatorname{Im}[S] = \underline{\underline{1 + \operatorname{Im}\left[\frac{1-(-1)^{N+1} e^{i(N+1)\theta}}{1+e^{i\theta}}\right]}}$$

4.

a) $B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi/2} 1\cdot \sin nx\,dx$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi/2} = \underline{\underline{\frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)}}$$

b) Odd 2π-periodic extension of f, midpoints of jumps:



c. $\displaystyle\sum_{n=1}^{\infty} B_n^2 = \underbrace{2\overset{0}{a_0^2} + \sum_{n=1}^{\infty}\left(\overset{0}{a_n^2} + \overset{B_n}{b_n^2}\right)}_{\text{from Full F.S. for } S(x)} \underset{\text{Parseval's}}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} S(x)^2\,dx$

$$= \frac{1}{\pi}\cdot\left(1^2\cdot\frac{\pi}{2} + 1^2\cdot\frac{\pi}{2}\right)$$

$$= \underline{\underline{1}}$$