Lesson 10 on 4.1, 4.2 ODE's

Lessons 7, 8 due tonight. 9, 10, 11 next Wed.
WebEx tonight 7:30–8:30 pm EDT

p. 345: 1. Compute $\mathbb{P}^{-1}$, $\hat{A} = \mathbb{P}^{-1} A \mathbb{P}$

a) Verify that $A, \hat{A}$ have same e-vals

b) Show that if $\vec{y}$ is an e-vect for $\hat{A}$, then $\mathbb{P}\vec{y}$ is an e-vect for A.

Linear const. coeff. ODE's

$$L[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \leftarrow \text{homog.}$$

Sol^n in $C(\mathbb{R}) \leftarrow$ vector space

Subspace Test: 1) y_1, y_2 sol^{n}s. Is $y_1 + y_2$?

$L[y_1] = 0$, $L[y_2]$ $\quad$ $L[y_1 + y_2]$

(Superposition principle.) $\quad = L[y_1] + L[y_2]$

$= 0 + 0$ ✓

2) y is a sol^n. Is cy?

$L[y] = 0$. $L[cy] = c\,L[y] = c \cdot 0 = 0$ ✓

Sol^{n}s space is a vect. space ✓

How to solve: Try $y = e^{rt}$:

$$a_n r^n e^{rt} + \cdots + a_1 r e^{rt} + a_0 e^{rt} \overset{\text{want}}{=} 0$$

$$\underbrace{(a_n r^n + \cdots + a_0)}_{P(r) \text{ must be } 0} \underset{\text{never } 0}{e^{rt}} = 0$$

$\boxed{P(r) = 0} \leftarrow$ Characteristic Eqn.

Case 1: Real root r_0 of mult 1.

Get $y = e^{r_0 t}$ sol^n.

Case 2: Real root r_0 of mult $m > 1$.

Get sol^{n}s $\underbrace{e^{r_0 t}, t e^{r_0 t}, \ldots, t^{m-1} e^{r_0 t}}_{m \text{ sol}^n\text{s.}}$

Case 3: $r = a \pm bi$ complex. Get two real sol^{n}s!

$e^{at} \cos bt$, $e^{at} \sin bt$

Why: $e^{(a+bi)t}$ is complex sol^n.

$e^{i\theta} = \cos\theta + i\sin\theta$
Euler

$$e^{at}e^{ibt} = e^{at}(\cos bt + i\sin bt)$$
$$= \underbrace{\left(e^{at}\cos bt\right)}_{u} + i\underbrace{\left(e^{at}\sin bt\right)}_{v}$$

complex sol^n

$$0 = L[u+iv] = \underbrace{L[u]}_{=0} + i\underbrace{L[v]}_{=0}$$

From one complex sol^n, get two real!

EX: $y^{(4)} + y'' = 0$

$$r^4 + r^2 = 0$$
$$r^2(r^2+1) = 0$$
$$r = \underbrace{0,0}_{e^{0t},\ te^{0t}},\ \pm i$$

$e^{0t},\ te^{0t}$
$1,\ t$

$0 \pm i$
$\begin{cases} e^{0t}\cos 1\cdot t \\ e^{0t}\sin 1\cdot t \end{cases}$
$\cos t,\ \sin t$

$$y = c_1 + c_2 t + c_3\cos t + c_4 \sin t$$

Case 4: $r = a \pm bi$ repeated m times [e.g. $(r^2+1)^5$]

$\pm i$ each have mult 5

$2m$ sol^{n}s $\begin{cases} e^{at}\cos bt,\ te^{at}\cos bt, \dots, t^{m-1}e^{at}\cos bt \\ e^{at}\sin bt,\ te^{at}\sin bt, \dots, t^{m-1}e^{at}\sin bt \end{cases}$

EX: $y^{(4)} + 2y'' + y = 0$ $\qquad (r^2+1)^2 = 0$

$\pm i$ double roots

$$y = c_1\cos t + c_2 \sin t + c_3 t\cos t + c_4 t \sin t$$

Q: How do we know this is the gen^l sol^n?

E & U Thm: $a_n y^{(n)} + \cdots + a_0 y = 0$

Initial Value Problem (IVP) $\begin{cases} y(0) = A_0 \\ y'(0) = A_1 \\ \vdots \\ y^{(n-1)}(0) = A_{n-1} \end{cases}$

A solution exists and it is unique.

$$\begin{cases} y(0) = c_1 y_1(0) + \cdots + c_n y_n(0) \overset{\text{want}}{=} A_0 \\ y'(0) = c_1 y_1'(0) + \cdots + c_n y_n'(0) = A_1 \\ \vdots \\ y^{(n-1)}(0) = c_1 y_1^{(n-1)}(0) + \cdots + c_n y_n^{(n-1)}(0) = A_{n-1} \end{cases}$$

$$\underbrace{\begin{bmatrix} y_1(0) & \cdots & y_n(0) \\ y_1'(0) & \cdots & y_n'(0) \\ y_1^{(n-1)}(0) & \cdots & y_n^{(n-1)}(0) \end{bmatrix}}_{\text{need } \det \neq 0\,!} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_{n-1} \end{pmatrix}$$

anything!

Def^n: Wronskian $= W[y_1, \ldots, y_n] = W = \det \left[\begin{array}{c} t\text{'s where} \\ 0\text{'s are} \end{array}\right]$

Fact: If $W(0) \neq 0$, then we can solve any and all IVP's. E & U Thm $\Rightarrow$ We have the gen^l sol^n.

Systems of ODEs: $\vec{x}' = A\vec{x}$

EX: $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. $\vec{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$

$$\begin{cases} \dfrac{dx_1}{dt} = 3x_1 + 5x_2 \\ \dfrac{dx_2}{dt} = 2x_1 - x_2 \end{cases} \qquad \vec{x}' = \begin{bmatrix} 3 & 5 \\ 2 & -1 \end{bmatrix} \vec{x}$$

How to solve: Try $\vec{x} = \vec{a} e^{\lambda t}$

Lesson 7 showed need: $\begin{cases} \det(A - \lambda I) = 0 \\ (A - \lambda I)\vec{a} = 0 \end{cases}$

$\begin{cases} \lambda \text{ eval} \\ \vec{a} \text{ e-vect for } \lambda \end{cases}$

Case: A $n \times n$ symm matrix.

Then A has real e-vals and a full set of evects, can be made orthonormal.

Get sol^{n}s: $\vec{x} = c_1 e^{\lambda_1 t}\vec{a}_1 + \cdots + c_n e^{\lambda_n t}\vec{a}_n$

Q: Is this the gen^l sol^n?

Equiv: Can we solve any and all IVP's?

IVP: $\vec{x}' = A\vec{x} \quad \vec{x}(0) = \vec{A}_0$

Need $\vec{x}(0) = c_1\vec{a}_1 + \cdots + c_n\vec{a}_n \overset{\text{want}}{=} \vec{A}_0$

$$\underbrace{\left[\vec{a}_1, \ldots, \vec{a}_n\right]}_{\text{need } \det \neq 0}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \vec{A}_0$$

anything!

Ha! $\vec{a}$'s unitary syst. So we have this.

$$\text{Wronskian} = \det\left[\vec{x}_1, \ldots, \vec{x}_n\right] =$$

$$= \det\left[\vec{a}_1 e^{\lambda_1 t}, \ldots, \vec{a}_n e^{\lambda_n t}\right]$$

$$= \underset{\text{never zero}}{e^{(\lambda_1 + \cdots + \lambda_n)t}} \underbrace{\det\left[\vec{a}_1, \ldots, \vec{a}_n\right]}_{\neq 0}$$

Facts: 1) If $W(0) \neq 0$, we have gen^l sol^n. ✓

2) $W(0) \neq 0 \iff W(t) \neq 0$ for all t.

EX: $y'' + 2y' + 3y = 0$

Can turn one 2nd order ODE into a 2x2 first order syst.

$$\begin{cases} x_1 = y \\ x_2 = y' \end{cases} \longleftarrow \text{The trick}$$

$$\begin{cases} x_1' = y' = x_2 \\ x_2' = y'' = \underline{-2x_2 - 3x_1} \end{cases}$$

$y'' = -2y' - 3y$

from ODE

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\vec{x}' = \underbrace{\mathbb{A}}_{2\times 2} \vec{x}$$

Trick

<u>3rd order:</u>

$$x_1 = y \qquad x_1' = y' = x_2$$
$$x_2 = y' \qquad x_2' = y'' = x_3$$
$$x_3 = y'' \qquad x_3' = y''' = \leftarrow \text{use ODE}$$

<u>Repeated root case:</u> $L[e^{rt}] = P(r)\,e^{rt}$

$$= (r-r_0)^m\, Q(r)\, e^{rt}$$

Clever trick: Diff w.r.t. r

$$L\Big[\underbrace{\frac{\partial}{\partial r} e^{rt}}_{te^{rt}}\Big] = \underbrace{m(r-r_0)^{m-1}\cdots}_{=\,0 \text{ if } r=r_0}$$

Finally, let $r = r_0$.