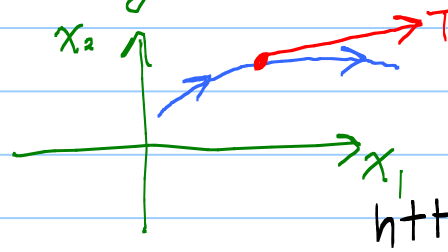
Lesson 11 on 4.3

9, 10, 11 due Wed.; 12, 13, 14 due Wed. 9/23
Exam 1 Thurs., Oct. 1 ← Crib sheet: 1 sheet, handwritten, both sides
No class Fri, Oct. 2
On campus: 8-9 pm WTHR 200

$$\begin{cases} \dfrac{dx_1}{dt} = F_1(x_1, x_2) \\[2ex] \dfrac{dx_2}{dt} = F_2(x_1, x_2) \end{cases}$$ ← no t on right "autonomous"

Geom meaning: Sol^n $\vec{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

$T = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \end{pmatrix}$



← phase plane

http://math.rice.edu/~dfield/dfpp.html

Linear case: $\begin{cases} \dfrac{dx_1}{dt} = 5x_1 - x_2 \\[2ex] \dfrac{dx_2}{dt} = 3x_1 + x_2 \end{cases}$ $\qquad \vec{x}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \vec{x}$

e-vals $\lambda = 2, 4$
e-vects $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\vec{x} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$

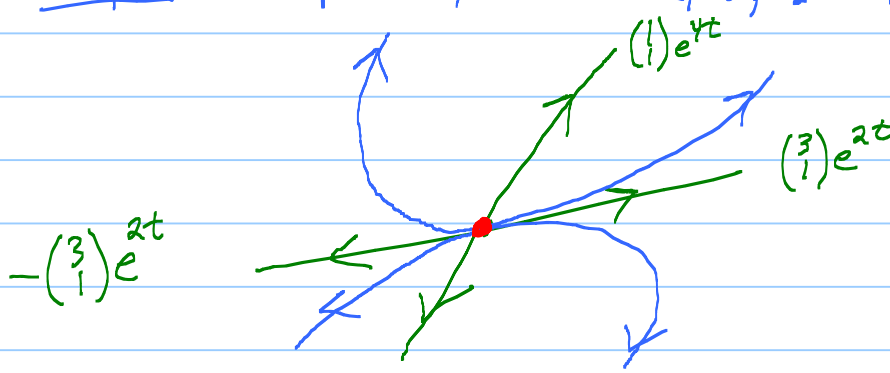
t big: way out
$t \to -\infty$: near origin

$$\vec{x} = e^{2t} \left[c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{(4-2)t} \right]$$

$t \to -\infty$: slides in along ± this dir. ; fade away

Near origin, sol^{n}s hug the e-vect. cor. to $\lambda = 2$ ← smaller one!

Step 1: Graph easy sol^{n}s: $c_1 = \pm 1,\ c_2 = 0$; $c_1 = 0,\ c_2 = \pm 1$



Improper Node ← Type
Unstable ← Stability

Near origin, traj. tangent to $\pm\binom{3}{1}$ dir.

Origin is a critical point.

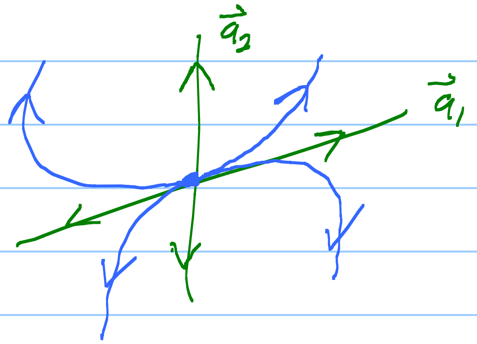
place where
$$\begin{cases} F_1(x_1,x_2)=0 \\ F_2(x_1,x_2)=0 \end{cases}$$

Method: $\vec{x}\,' = \mathbb{A}\vec{x}$, $\mathbb{A}$ 2x2

Get e-vals λ_1, λ_2 and e-vects.

Case 1: $0<\lambda_1<\lambda_2$ real, >0, $\neq$

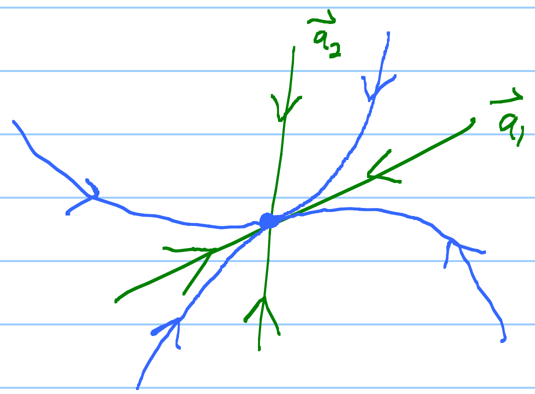
closest to zero



Imp. Node.
Unstable

Case 2: $\lambda_2<\lambda_1<0$ real, <0, $\neq$

closest to zero

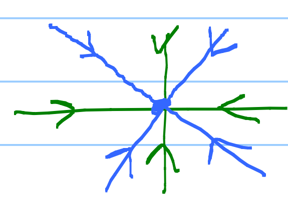


Imp. Node
Asymptotically stable!

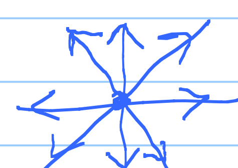
Node rule: Near the origin, traj. hug the ±e-vect dir cor to e-val closest to zero.
(Far away, they look parallel to the other e-vect.)

Case $\lambda_1=\lambda_2$, real $\neq 0$

i) Proper Node Case
Full set of e-vects.



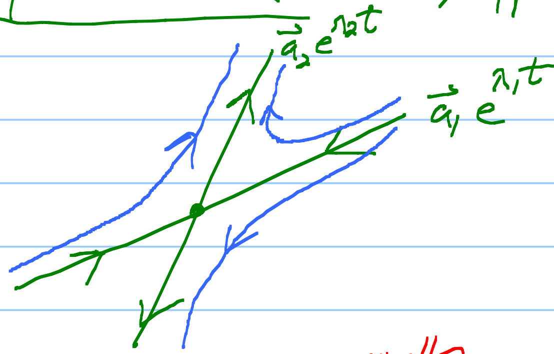
$\lambda<0$



$\lambda>0$

ii) Defective case. Ugh!

Case $\lambda_1 < 0 < \lambda_2$ real, opp. sign



Saddle Point
Unstable

$t \to -\infty$: big, small

$$\vec{x} = c_1 \vec{a}_1 e^{\lambda_1 t} + c_2 \vec{a}_2 e^{\lambda_2 t}$$

$t \to \infty$: small, big

Case $\lambda = a \pm bi$: Complex

For $\lambda = a + bi$, get complex e-vect. $\vec{A} + i\vec{B}$

$$\vec{x} = (\vec{A} + i\vec{B}) e^{(a+bi)t} = (\vec{A} + i\vec{B})(e^{at}\cos bt + i e^{at}\sin bt)$$

$$= \left[\vec{A}e^{at}\cos bt - \vec{B}e^{at}\sin bt\right] + i\left[\vec{A}e^{at}\sin bt + \vec{B}e^{at}\cos bt\right]$$

$i^2 = -1$

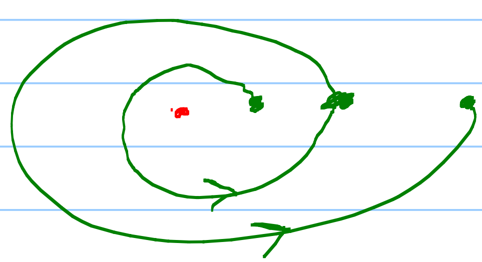
two real sol^{n}s!

Start fresh: Gen^l Sol^n

$$\vec{x} = c_1 e^{at}(\vec{A}\cos bt - \vec{B}\sin bt) + c_2 e^{at}(\vec{A}\sin bt + \vec{B}\cos bt)$$

Real! Sol^n

Subcase: $a > 0$ e^{at} grows. Rest is periodic



Spiral (out)
Unstable

Subcase: $a<0$ e^{at} shrinks

Spiral (in)
Asymp. Stable

Subcase: $a=0$

Center
Stable ← Just stable.
Not asymp. stable

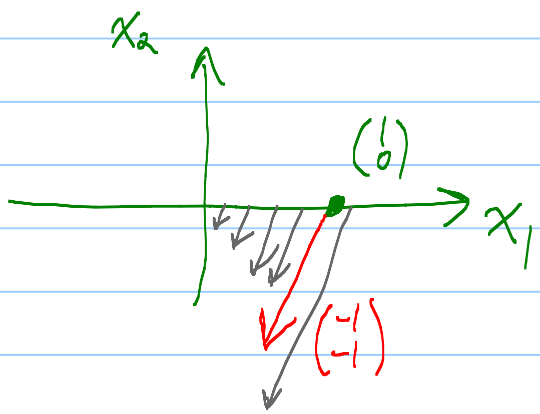
Q: Clockwise or counterclockwise?
CW CCW

EX: $\vec{x}\,' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \vec{x}$ $\lambda = -1 \pm i$

$a = -1 < 0$
Spiral in.
Asympt. Stable

CW or CCW?
Test field at $k\begin{pmatrix}1\\0\end{pmatrix}$.

At $\begin{pmatrix}1\\0\end{pmatrix}$: $\vec{x}\,' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} k\begin{pmatrix}1\\0\end{pmatrix} = k\begin{pmatrix}-1\\-1\end{pmatrix}$



Clockwise