1

Lesson 12 on 4.4

WebEx Off. Hrs. this week: T, W 7:30–8:30

EX: $\vec{x}\,' = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}\vec{x}$

$\det(A-\lambda I) = (-2-\lambda)^2 + 4 = 0$

$(-2-\lambda)^2 = -4$

$-2-\lambda = \pm 2i$

$\lambda = -2 \pm 2i \leftarrow$ complex

$\uparrow_{Re<0}$ $\left\{\begin{array}{l}\text{Spiral in} \\ \text{Asymp. Stable}\end{array}\right.$

attracting

For $\lambda = -2+2i$: $(A-\lambda I)\vec{a} = \vec{0}$, $\lambda = -2+2i$

$$\left[\begin{array}{cc|c} -2-(-2+2i) & 2 & 0 \\ -2 & -2-(-2+2i) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right] \quad R_2 = -i R_1$$

$$\sim \left[\begin{array}{cc|c} \overset{A}{-2i} & \overset{B}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} A & B & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\begin{pmatrix} B \\ -A \end{pmatrix}$ or $\begin{pmatrix} -B \\ A \end{pmatrix}$

$\begin{pmatrix} 2 \\ 2i \end{pmatrix}$ or

even better $\begin{pmatrix} 1 \\ i \end{pmatrix}$

Complex Sol^n: $\begin{pmatrix} 1 \\ i \end{pmatrix}\left(e^{-2t}\cos 2t + i e^{-2t}\sin 2t\right)$

$$e^{-2t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]\left(\cos 2t + i\sin 2t\right)$$

$$= e^{-2t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos 2t \overset{i^2=-1}{-} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin 2t\right] + i e^{-2t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin 2t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos 2t\right]$$

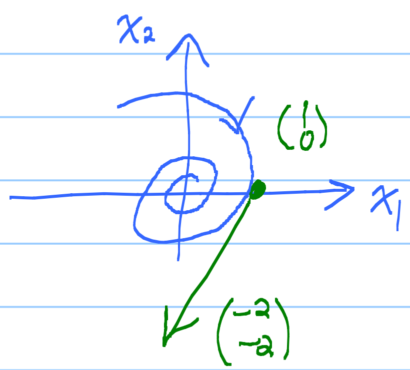
$$= \underbrace{\left[\begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^{-2t}\right]}_{\vec{x}_1} + i \underbrace{\left[\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^{-2t}\right]}_{\vec{x}_2}$$

$\vec{x}_1$ and $\vec{x}_2$ are real sol^{n}s.

Aha! $W[\vec{x}_1, \vec{x}_2](0) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq 0.$

Gen^l Sol^n $\vec{x} = c_1 e^{-2t} \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}$

CW or CCW? $\vec{x}\,'$ at $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$



Spiral in
CW

Last case: λ repeated e-val $\neq 0$, but only one e-vect. [A defective, geom mult < alg mult]

Get one sol^n: $\vec{x}_1 = \vec{a}\, e^{\lambda t}$.

Need a second sol^n: Hmmm. Try $t \vec{a} e^{\lambda t}$. Nope!

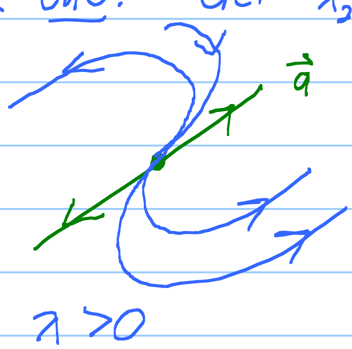
Better try: $\vec{x}_2 = \underset{\uparrow \text{ e-vect.}}{\vec{a}}\, t e^{\lambda t} + \underset{\uparrow\, \vec{b} \text{ a const. vect.}}{\vec{b}}\, e^{\lambda t}$

Plug in and figure out conditions on $\vec{b}$.

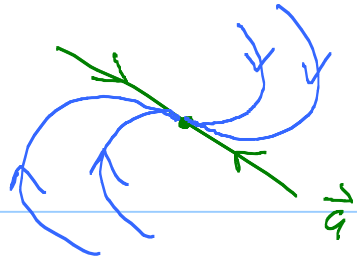
Get $\underbrace{(A - \lambda I)}_{\det = 0!} \vec{b} = \vec{a}$

Linear Alg fact. This has ∞ many sol^{n}s $\vec{b}$.

Pick one. Get $\vec{x}_2$.



Improper Node
Unstable



$\lambda < 0$

Improper Node
Asympt. stable

$\left(te^{-kt} \to 0 \text{ as } t \to \infty, \; k > 0 \right)$

Non-linear systems: How to linearize a C^2-smooth fcn.

$$f(t) \cong f(t_0) + f'(t_0)(t - t_0) \quad \text{near } t_0$$

Taylor's Thm

$$u(x,y) = u(x_0,y_0) + \frac{\partial u}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial u}{\partial y}(x_0,y_0)(y-y_0) + R(x,y)$$

super small near (x_0,y_0)

$$\text{where } \lim_{(x,y)\to(x_0,y_0)} \frac{R(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

Non-linear system:
$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases}$$

Crt. Pt. (x_0,y_0):
$$\begin{cases} f(x_0,y_0) = 0 \\ g(x_0,y_0) = 0 \end{cases}$$

Linearized System at (x_0,y_0): $\vec{x}\,' = \mathbb{A}\vec{x}$ where

$$\mathbb{A} = \left.\begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{bmatrix}\right|_{(x_0,y_0)}$$

New vars:
$\tilde{x} = (x - x_0)$
$\tilde{y} = (y - y_0)$

Critical Pt. becomes new origin of linearized syst.

EX: Competing species: x = deer, y = elk on island.

$$\begin{cases} \dfrac{dx}{dt} = x - x^2 - xy \\ \dfrac{dy}{dt} = \frac{1}{2}y - \frac{1}{4}y^2 - \frac{3}{4}xy \end{cases}$$

Step 1: Find crt. pts.

$$\begin{cases} x(1-x-y) = 0 & (A) \\ y\left(\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x\right) = 0 & (B) \end{cases}$$

For (A) to hold, need $x=0$ or $(1-x-y)=0$

Case 1: $x=0$

(B) $y\left(\frac{1}{2} - \frac{1}{4}y - 0\right) = 0$

Need $y=0$ or $\frac{1}{2} - \frac{1}{4}y = 0$

$y=2$

$$\boxed{(0,0),\ (0,2)}$$

Case: $1-x-y=0$ (A) ✓

To make (B) hold, need $y=0$ or $\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x = 0$

Subcase: $y=0$. (A): $1-x-0=0$ $x=1$.

$$\boxed{(1,0)}$$

Subcase: $\begin{cases} \frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x = 0 & (B) \\ 1 - x - y = 0 & (A) \end{cases}$

Get $\boxed{\left(\frac{1}{2}, \frac{1}{2}\right)}$

Step 2: Linearize

$$A = \begin{bmatrix} \frac{\partial}{\partial x}(x - x^2 - xy) & \frac{\partial}{\partial y}(x - x^2 - xy) \\ \frac{\partial}{\partial x}\left(\frac{1}{2}y - \frac{1}{4}y^2 - \frac{3}{4}xy\right) & \frac{\partial}{\partial y}\left(\frac{1}{2}y - \frac{1}{4}y^2 - \frac{3}{4}xy\right) \end{bmatrix}\Bigg|_{(x_0, y_0)}$$

$$= \begin{bmatrix} 1 - 2x - y & -x \\ -3/4\, y & \frac{1}{2} - \frac{1}{2}y - \frac{3}{4}x \end{bmatrix}\Bigg|_{(x_0, y_0)}$$

Step 3: Look at each crt. pt.

At $(0,0)$: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$ $\lambda = \frac{1}{2}, 1$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Imp. Node
Unstabe

At $(0,2)$: $A = \begin{bmatrix} -1 & 0 \\ -3/2 & -1/2 \end{bmatrix}$ $\lambda = -\frac{1}{2}, -1$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Imp. Node
A. Stable

At $(1,0)$: $A = \begin{bmatrix} -1 & -1 \\ 0 & -1/4 \end{bmatrix}$ $\lambda = -\frac{1}{4}, -1$

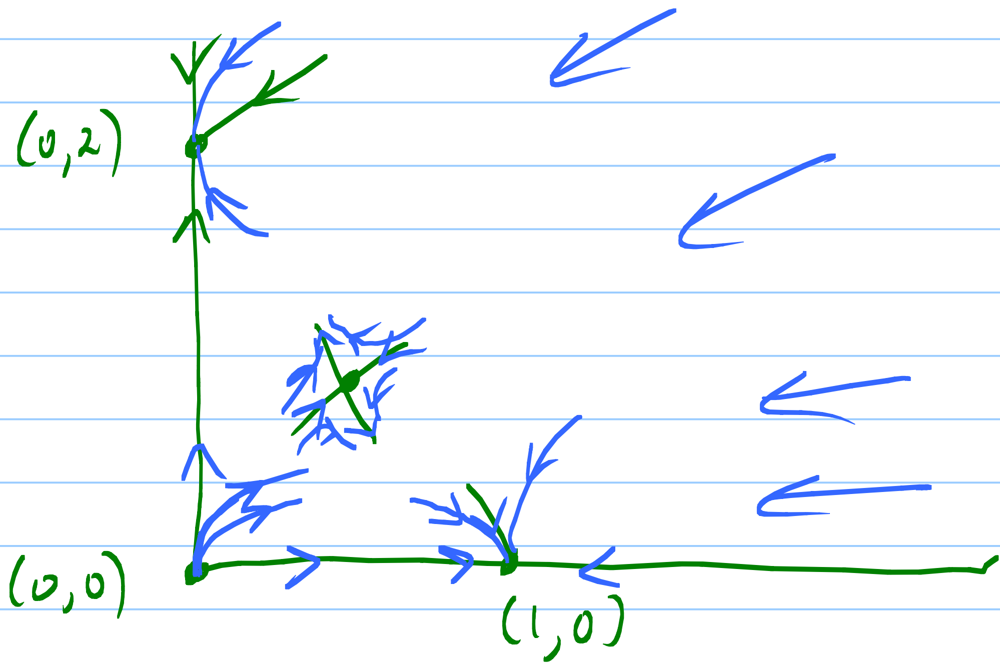
$\begin{pmatrix} 4 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

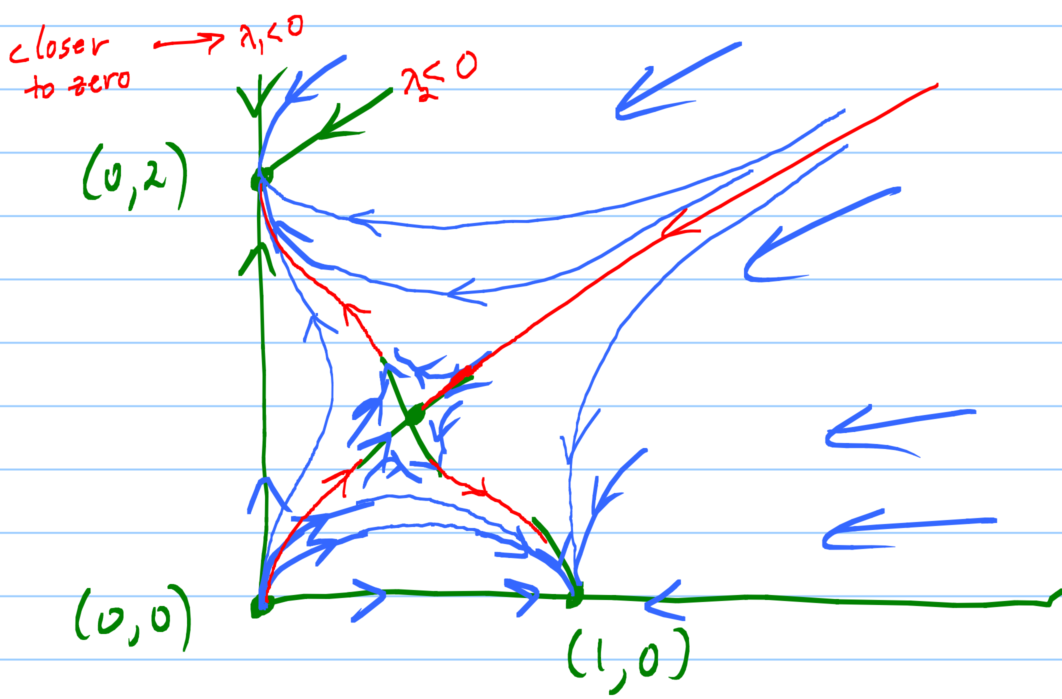
Imp. Node
A. Stable

At $\left(\frac{1}{2}, \frac{1}{2}\right)$: $A = \begin{bmatrix} -1/2 & -1/2 \\ -3/8 & -1/8 \end{bmatrix}$ $\lambda = \frac{-5 \pm \sqrt{57}}{16}$

$\begin{pmatrix} 1 \\ \frac{-3 \pm \sqrt{57}}{8} \end{pmatrix}$

Saddle
Unstable


(0,2)
(0,0)
(1,0)


closer to zero
$\lambda_1 < 0$
$\lambda_2 < 0$
(0,2)
(0,0)
(1,0)