Lesson 18 on 6.1 Laplace Transform

18, 19, 20 due Wed. Oct. 14

$$\mathcal{L}[\underbrace{f(t)}_{t\geq 0}] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

Huge Fact 1: $\boxed{\mathcal{L}[f'(t)] = sF(s) - f(0)}$

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f'(t)] - f'(0)$$

$$\boxed{\mathcal{L}[f''(t)] = s^2F(s) - sf(0) - f'(0)}$$

Huge Fact 2: If $\mathcal{L}[f_1] = \mathcal{L}[f_2]$ for $s > M$, some M

Then $f_1 \equiv f_2$! Can invert $\mathcal{L}$!

Master plan: ODE $y'' + y = f(t)$ $\begin{cases} y(0) = y_0 \\ y'(0) = y_0' \end{cases}$

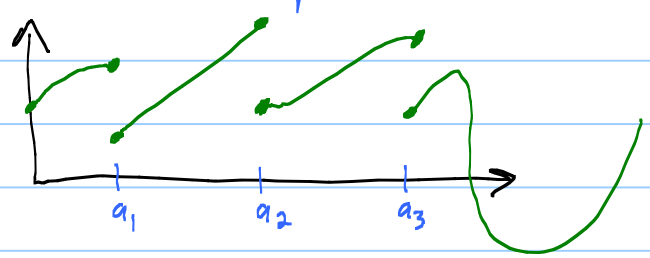
Hit ODE with $\mathcal{L}$: $(s^2 Y - sy_0 - y_0') + Y = F$

Use algebra to get Y. Undo $\mathcal{L}$ to get y

$$\mathcal{L}[\text{ODE}] \longrightarrow \text{Algebra} \xrightarrow{\mathcal{L}^{-1}} \text{sol}^{n}$$

Requirements for $\mathcal{L}[f]$ to make sense:

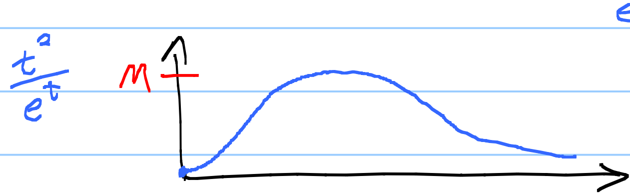
1) f needs to be "piecewise continuous" on $[0, \infty)$



2) f needs to be of "exponential type"
meaning there are positive constants
M and K such that $|f(t)| \leq Me^{Kt}$ for $t \geq 0$.

EX: $t^2 e^{3t} \cos 2t$

$$\frac{t^2}{e^t} \to 0 \text{ as } t \to \infty.$$



$$\frac{t^2}{e^t} \leq M$$

$$t^2 \leq Me^t$$

$$\left| t^{a} e^{3t} \cos 2t \right| \le M e^{t} \cdot e^{3t} \cdot 1 \le M e^{4t} \checkmark$$

$$\int e^{kt}\,dt = \frac{1}{k} e^{kt} + C$$

EX: $e^{(e^t)}$ is not of exp type!

Start our table: $\mathcal{L}[e^{at}] = \int_0^\infty e^{-st} \left[e^{at} \right] dt$

$$= \int_0^\infty e^{(a-s)t}\,dt = \lim_{B\to\infty} \int_0^B e^{(a-s)t}\,dt$$

$$= \lim_{B\to\infty} \left[\frac{1}{a-s} e^{(a-s)t} \right]_0^B$$

$$= \lim_{B\to\infty} \left[\frac{1}{a-s} e^{(a-s)B} - \frac{1}{a-s} e^{0} \right]$$

Need $a-s<0$, $\boxed{s>a}$ ← Take this

$$= 0 - \frac{1}{a-s} = \frac{1}{s-a}$$

$f(t)$	$F(s)$
e^{at}	$\frac{1}{s-a}$, $s>a$
$\sin bt$	$\frac{b}{s^2+b^2}$, $s>0$
$\cos bt$	$\frac{s}{s^2+b^2}$, $s>0$
1	$\frac{1}{s}$, $s>0$
t	$\frac{1}{s^2}$, $s>0$
t^n	$\frac{n!}{s^{n+1}}$, $s>0$
$\sinh bt$	$\frac{b}{s^2-b^2}$, $s>b$

$$\mathcal{L}[\sinh bt] = \mathcal{L}\left[\tfrac{1}{2}\left(e^{bt} - e^{-bt}\right)\right]$$

$$= \frac{1}{2}\mathcal{L}[e^{bt}] - \frac{1}{2}\mathcal{L}[e^{-bt}]$$

$$= \frac{1}{2}\left(\underset{\uparrow\, s>b}{\frac{1}{s-b}} - \underset{\uparrow\, s>(-b)}{\frac{1}{s-(-b)}}\right) = \frac{b}{s^2-b^2}$$

$\leftarrow$ need $s>b$ to have both

<u>The s-shifting rule</u>: If $\mathcal{L}[f(t)] = F(s)$

then $\mathcal{L}[e^{at} f(t)] = F(s-a)$

<u>Why</u>: $$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= \int_0^\infty e^{-st}\left[e^{at} f(t)\right]dt \quad \checkmark$$

$f(t)$	$F(s)$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos bt$	$\frac{(s-a)}{(s-a)^2+b^2}$

<u>Tricks</u>: Partial fractions, comp. square

<u>Completing the square</u>: As^2+Bs+C

<u>Step 1</u>: Factor out A: $A\Big(\underbrace{s^2+\frac{B}{A}s+\frac{C}{A}}_{s^2+bs+c}\Big)$

<u>Step 2</u>: $$s^2+bs+c = (s+\tfrac{b}{2})^2 + (c-\tfrac{b^2}{4})$$

$\frac{1}{2}b$; clean up

<u>Prob</u>: Find $\mathcal{L}^{-1}\left[\frac{2s}{s^2-4s+13}\right]$ $\leftarrow$ complex roots. Comp. Square

$$s^2-4s+13 = (s-2)^2+(13-4) = (s-2)^2+3^2$$

$\frac{1}{2}(-4)$

$$\frac{2s}{s^2-4s+13} = \frac{2\left[\overbrace{(s-2)+2}^{s}\right]}{(s-2)^2+3^2}$$

$$= 2\,\frac{(s-2)}{(s-2)^2+3^2} + \frac{4}{3}\cdot\frac{3}{(s-2)^2+3^2}$$

$$= \mathcal{L}\left[\underbrace{2e^{2t}\cos 3t + \frac{4}{3}e^{2t}\sin 3t}_{\text{answer}}\right]$$

Better table:

$f(t)$	$F(s)$
$\ell\ell\ell$	$\frac{s}{(s-a)^2+b^2}$
$\ell\ell\ell$	$\frac{1}{(s-a)^2+b^2}$

EX: $\frac{s+4}{s^2+4s+3} = \frac{s+4}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$

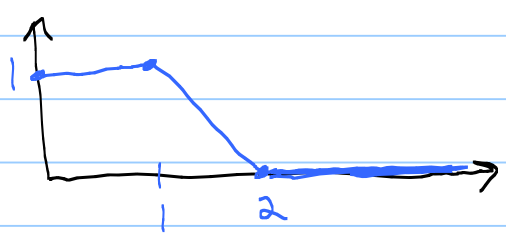
real roots

partial fractions

(Get A, B)

$$= \mathcal{L}\left[Ae^{-t} + Be^{-3t}\right]$$

Virtue of $\mathcal{L}$: Hardly notices piecewise definedness!



$$f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \\ 0 & t \ge 2 \end{cases}$$

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}\cdot 1\,dt + \int_1^2 e^{-st}(2-t)\,dt + \int_2^\infty e^{-st}\cdot 0\,dt$$

$$= \frac{1}{s} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2}$$

Integration by parts:

$$I = \int \underset{u}{t}\, \underbrace{e^{-st}\, dt}_{dv}$$

$$\left[\begin{array}{ll} u = t & dv = e^{-st}\,dt \\ du = dt & v = \int e^{-st}\,dt \leftarrow \text{no } + C \\ & v = -\frac{1}{s}e^{-st} \end{array}\right.$$

$$= uv - \int v\, du = (t)\left(-\frac{1}{s}e^{-st}\right) - \int\left(-\frac{1}{s}e^{-st}\right)dt$$

$$= -\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st} + C$$

$$\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du$$