Lesson 19 on 6.2 WebEx Off. Hr. this week Thurs. 8-9 pm

Office Hrs. T, W 2:00-3:00 in MATH 750

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

Why: $\displaystyle\int_0^B \underbrace{e^{-st}}_{u}\ \underbrace{f'(t)\,dt}_{dV}$

$$\left[\begin{array}{ll} u = e^{-st} & dv = f'(t)dt \\ du = -se^{-st}dt & v = \int f'(t)dt = f(t) \end{array}\right.$$

$$= uv\Big|_0^B - \int_0^B v\,du$$

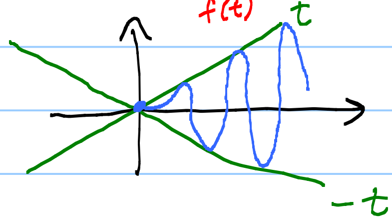
$$= \underbrace{\left(e^{-st}\right)}_{u}\cdot\underbrace{\left(f(t)\right)}_{v}\Big|_0^B - \int_0^B \underbrace{f(t)}_{v}\ \underbrace{\left(-se^{-st}dt\right)}_{du}$$

$$= \underbrace{e^{-sB}\underbrace{f(t)}_{|f(t)|\le Me^{kt}}}_{\substack{\text{Key: Need } s>k \\ \text{for } e^{-sB} \text{ to kill } f(t) \\ \text{as } B\to\infty}} - 1\cdot f(0) + s\underbrace{\int_0^B e^{-st} f(t)\,dt}_{\xrightarrow[B\to\infty]{} \mathcal{L}[f]}$$

$$\xrightarrow[B\to\infty]{} 0 - f(0) + s\mathcal{L}[f] \quad \text{provided } s>k.$$

Table entry: $\mathcal{L}[f^{(n)}(t)] = s^nF(s) - f(0)s^{n-1} - f'(0)s^{n-2} - \cdots - f^{(n-1)}(0)$

EX: $\mathcal{L}[\underbrace{t\sin t}_{f(t)}] = F(s) \leftarrow f(0)=0$



Resonance!

$t\sin t$

Trick: $f'(t) = t\cos t + \sin t \leftarrow f'(0)=0$

$f''(t) = -\underbrace{t\sin t}_{f(t)} + 2\cos t$

$$\mathcal{L}[f''(t)] = \begin{array}{l} -\mathcal{L}[\overbrace{t\sin t}^{f(t)}] + 2\dfrac{s}{s^2+1^2} \\ s^2F(s) - s\underbrace{f(0)}_{0} - \underbrace{f'(0)}_{0} \end{array} \Big] \text{ equate}$$

$$\boxed{-F(s) + \frac{2s}{s^2+1} = s^2 F(s)}$$ ← solve for F

$$\mathcal{L}[t\sin t] = F(s) = \frac{2s}{(s^2+1)^2}$$

Fact: $$\mathcal{L}\Big[\underbrace{\int_0^t f(u)\,du}_{g(t)}\Big] = \frac{1}{s}F(s)$$

Fund Thm Calc: $g'(t) = f(t)$

Why: $F(s) = \mathcal{L}[f] =$ ← $f = g'$

$$= \mathcal{L}[g'] = s\,\mathcal{L}[g] - \underbrace{g(0)}_{0}$$

$g(0) = \int_0^0 f(t)\,dt = 0$

Get $\mathcal{L}[g] = \frac{1}{s}F(s)$ ✓

EX: $$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = ?$$

$$\frac{1}{s}\cdot\underbrace{\frac{1}{s^2+1}}_{F(s)} = \mathcal{L}\left[\int_0^t \sin u\,du\right]$$

$f(t) = \sin t$

$$= \mathcal{L}\left[(-\cos u)\Big|_0^t\right]$$

$$= \mathcal{L}\left[-\cos t - (-\cos 0)\right]$$

$$= \mathcal{L}[\underbrace{1-\cos t}_{\text{ans.}}]$$

EX: Another way:

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

↖ partial fractions decomposition

Step 1: Multiply by denom: $s(s^2+1)$

$$1 = A(s^2+1) + (Bs+C)s$$

Step 2: Collect coeff of powers of s

$$0\cdot s^2 + 0\cdot s + 1 = (A+B)s^2 + (C)s + (A)$$

Step 3: Equate coeff

$A+B=0 \leftarrow B=-A=-1$

$C=0$

$A=1 \leftarrow A=1$

$$\begin{cases} A=1 \\ B=-1 \\ C=0 \end{cases}$$

Done: $$\frac{1}{s(s^2+1)} = \frac{\overset{A}{(1)}}{s} + \frac{\overset{B}{(-1)}s + \overset{C}{(0)}}{s^2+1}$$

$$= \frac{1}{s} - \frac{s}{s^2+1}$$

$$= \mathcal{L}[\underbrace{1 - \cos t}_{\text{ans.}}]$$

Partial Fractions: $$\frac{s^4 + 5s^3 + s + 1}{s^2(s-1)^3(s-4)\underbrace{(s^2+1)}_{\text{complex roots}}\underbrace{(s^2+2s+2)^2}_{\text{comp roots}}}$$

deg Num $<$ deg Denom

If not, do long division first

$$= \underbrace{\frac{A}{s^2} + \frac{B}{s}}_{\text{for } s^2 \text{ term}} + \underbrace{\frac{C}{(s-1)^3} + \frac{D}{(s-1)^2} + \frac{E}{(s-1)}}_{\text{for } (s-1)^3 \text{ term}} + \underbrace{\frac{F}{(s-4)}}_{\text{for } (s-4) \text{ term}}$$

$s = (s-0)$

$$+ \underbrace{\frac{Gs+H}{(s^2+1)}}_{\text{for } (s^2+1) \text{ term}} + \underbrace{\frac{Is+J}{(s^2+2s+2)^2} + \frac{Ks+L}{(s^2+2s+2)}}_{\text{for } (s^2+2s+2) \text{ term}}$$

Real Problem: $y'' + 4y = e^{2t}$ $\begin{cases} y(0)=1 \\ y'(0)=0 \end{cases}$

Think: Circuit: e^{2t} input, $y=$ output

Spring mass: e^{2t} forcing, $y=$ response

Hit ODE with $\mathcal{L}$: $\mathcal{L}[y''] + 4\mathcal{L}[y] = \mathcal{L}[e^{2t}]$

$$\left(s^2 Y - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{0}\right) + 4Y = \frac{1}{s-2}$$

$$(s^2+4)Y = s + \frac{1}{s-2}$$

$$Y = \underbrace{\frac{1}{s^2+4}}_{\text{Transfer function}} \left[s + \frac{1}{s-2} \right]$$

$$Y = \underbrace{\frac{s}{s^2+4}}_{p1} + \underbrace{\frac{1}{(s-2)(s^2+4)}}_{p2}$$

$$p2: \frac{1}{(s-2)(s^2+4)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+4}$$

Step 1: $1 = A(s^2+4) + (Bs+c)(s-2)$

Step 2: $1 = \underbrace{(A+B)}_{0}s^2 + \underbrace{(C-2B)}_{0}s + \underbrace{(4A-2C)}_{1}$

Step 3:

$$A+B=0$$
$$C-2B=0$$
$$4A-2C=1$$

$$\begin{array}{ccc|c} A & B & C & \\ \hline 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 4 & 0 & -2 & 1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/8 \\ 0 & 1 & 0 & -1/8 \\ 0 & 0 & 1 & -1/4 \end{array}\right]$$

$A = 1/8$
$B = -1/8$
$C = -1/4$

$$Y = \underbrace{\frac{s}{s^2+4}}_{(p1)} + \underbrace{\frac{1/8}{s-2} + \frac{(-\frac{1}{8})s + (-\frac{1}{4})}{s^2+4}}_{(p2)}$$

$$Y = \frac{7}{8} \cdot \frac{s}{s^2+2^2} + \frac{1/8}{s-2} - \frac{1}{4} \cdot \frac{1}{2} \frac{2}{s^2+2^2}$$

$$y = \frac{7}{8} \cos 2t + \frac{1}{8} e^{2t} - \frac{1}{8} \sin 2t$$

HWK hints: p. 216: 17

$$f(t) = te^{at}$$
$$f'(t) = ate^{at} + e^{at}$$

$\mathcal{L}[f'] =$ One way / Another way — equate

p. 216 : 18.

$f(t) = \sin^2 \omega t$

$f'(t) = \omega 2 \sin \omega t \cos \omega t = \omega \sin 2\omega t$

$\mathcal{L}[f'] =$ two ways.