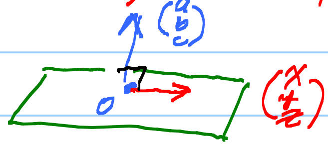
Lesson 2 MA 527 (Overflow room WANG 2563)

$A\vec{x} = \vec{0}$ ← homogeneous (always has $\vec{x} = \vec{0}$ sol^n)

$A\vec{x} = \vec{b}$ ← non-homog (might have no sol^{n}s, or ∞ many sol^{n}s, or a unique sol^n)

3-D case: $ax + by + cz = 0$

$(a,b,c) \cdot (x,y,z)$



See p. 274 for 3-D pics.

Gaussian elimination:

$$\begin{cases} 2x_1 + x_2 - x_3 = 3 \\ x_1 - x_2 + 2x_3 = 1 \\ -4x_1 + 6x_2 - 7x_3 = 1 \\ 2x_1 \qquad + x_3 = 3 \end{cases}$$

"overdetermined" too many eqns.

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 1 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{array}\right] = A^{\#}$$

$\underbrace{\qquad}_{A}$ $\vec{b}$

HS rules:
1) Can swap two eqns.
2) Can multiply an eqn by $c \neq 0$.
3) Can multiply an eqn by $c \neq 0$ and add it to another.

Goal: Eliminate vars.

College rules: Replace eqn by row.

Goal: Reduce to row echelon form, and do back substitution.

EX:

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 8 \\ 0 & 5 & 6 & 9 \\ 0 & 0 & 7 & 10 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

← dangerous spot!

$2x_1 + 3x_2 + 4x_3 = 8$ E1

$5x_2 + 6x_3 = 9$ E2

$7x_3 = 10$ E3

$0 = 0$ ← whew!

Back sub: E3: $x_3 = 10/7$ ✓

E2: $5x_2 + 6\left(\frac{10}{7}\right) = 9$ ← solve for x_2.

E1: $2x_1 + 3(\text{know}) + 6\left(\frac{10}{7}\right) = 8$ ← solve for x_1

Signature of an inconsistent sys (no sol'ns).

$$\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 3 \\ 0 & 0 & & 0 & 0 \end{array}\right] \leftarrow \text{ouch} \qquad 0 = 3 \ ↯.$$

EX: $\begin{cases} x_1 - x_2 + x_3 - 2x_4 = -1 \\ 2x_1 - 2x_2 + x_3 - 2x_4 = -3 \\ -x_1 + x_2 - 2x_3 + 4x_4 = 0 \end{cases}$ "underdetermined" too many vars.

$$\begin{array}{cccccc} & & & & & \\ \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] & \leftarrow \text{whew!} \\ \uparrow \ x_1 \text{ col} \qquad \uparrow \ x_3 \text{ col} & \end{array}$$

x_1, x_3 are bound var.

Others are free. x_2, x_4 are free vars.

If free vars exist, there are ∞ many sol'ns.

Method. Step 1: Set free vars = to parameters.

$\begin{cases} x_2 = t_1 \\ x_4 = t_2 \end{cases}$ free to vary.

Step 2: Plug t's into (row reduced) eqns and solve for bound vars.

$$\begin{cases} x_1 - \underbrace{\boxed{t_1}}_{x_2} = -2 & \text{E1} \\ x_3 - 2\underbrace{\boxed{t_2}}_{x_4} = 1 & \text{E2} \end{cases}$$

E1: $x_1 = -2 + t_1$

E2: $x_3 = 1 + 2t_2$

Step 3: Make a list of all vars.

$$\begin{cases} x_1 = -2 + t_1 \\ x_2 = t_1 \\ x_3 = 1 + 2t_2 \\ x_4 = t_2 \end{cases}$$

Step 4: Pull it apart

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \underbrace{\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\substack{\text{constants} \\ \text{point in } \mathbb{R}^4}} + \underbrace{t_1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}}_{\text{"2-D plane in } \mathbb{R}^4\text{"}}$$

$$= \underbrace{\vec{v}_p}_{\substack{\text{a particular} \\ \text{sol}^n \text{ to} \\ A\vec{x} = \vec{b}}} + \underbrace{\left(t_1 \vec{v}_1 + t_2 \vec{v}_2\right)}_{\substack{\text{gen}^l \text{ sol}^n \\ \text{to} \\ A\vec{x} = \vec{0}}}$$

Facts: $A\vec{x} = \vec{b}$, $A^\# = [A \mid \vec{b}] \rightsquigarrow [E \mid \vec{e}]$

Rank(A) = # non-zero rows in E = (# bound vars)

Rank$(A^\#)$ = " " $[E \mid \vec{e}]$

1) Rank(A) < Rank$(A^\#)$ ⟸ no sol^{n}s!

2) Rank(A) = Rank$(A^\#)$.

a) Rank(A) = #vars. All vars bound.

unique sol^n

b) Rank(A) < #vars. Free vars exist.

∞ many sol^{n}s!

Homog case: $\vec{b} = 0$. Case 1) cannot happen.