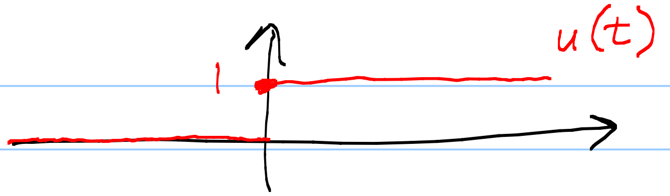
Lesson 20 on 6.3

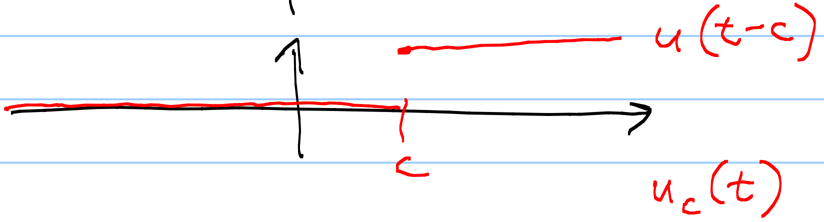
No lecture Monday.
18,19,20 due Wed

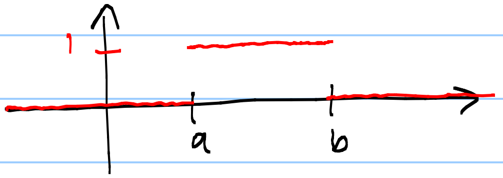
Piazza Tues
WebEx Wed 8-9pm

Unit step function (Heaviside fcn)



Turn on at time c





Switch: $\underset{\text{on at } a}{\underset{\uparrow}{u(t-a)}} - \underset{\text{off at } b}{\underset{\uparrow}{u(t-b)}}$

Two big facts: 1) s-shifting formula: $\mathcal{L}[e^{at}f(t)] = F(s-a)$

2) t-shifting formula: $\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}F(s)$

Take $f(t) \equiv 1$: $\mathcal{L}[u(t-c)] = e^{-cs}\frac{1}{s}$

Why: $\mathcal{L}[u(t-c)f(t-c)] = \int_0^\infty e^{-st}\underbrace{u(t-c)}_{\substack{0 \quad t<c \\ 1 \quad t>c}} f(t-c)\,dt$

$I = \int_c^\infty e^{-st}\cdot 1 \cdot f(\underbrace{t-c}_{\tau})\,dt$ $\qquad \tau = t-c \leftarrow \underline{\underline{t = \tau + c}}$

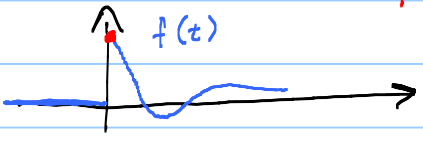
$d\tau = dt$

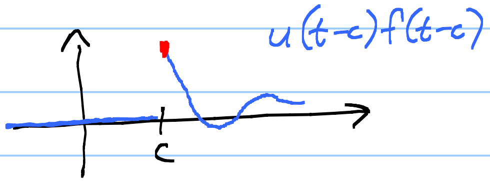
When $t = c$: $\tau = c - c = 0$

as $t \to \infty$: $\tau \to \infty$ too

$$I = \int_{\tau=0}^\infty \underbrace{e^{-s(\tau+c)}}_{e^{-s\tau}e^{-cs}} f(\tau)\,d\tau$$

$$= \underbrace{e^{-cs}\int_{\tau=0}^\infty e^{-s\tau} f(\tau)\,d\tau}_{F(s)} \quad \checkmark$$





Trick: $t = [(t-c)+c]$

EX: $\mathcal{L}[u(t-3)\underset{\uparrow \text{wish } f(t-3)}{t^2}] = ?$

$$u(t-3)[\overbrace{(t-3)+3}^{t}]^2 = u(t-3)\underbrace{(t-3)^2} + 6u(t-3)\underbrace{(t-3)} + 9u(t-3)$$

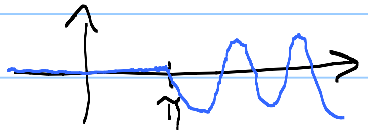
$f(t-3)=(t-3)^2$ $\quad$ $g(t-3)=t-3$

$f(t)=t^2$ $\quad$ $g(t)=t$

$$\mathcal{L} = e^{-3s}F(s) + 6e^{-3s}G(s) + 9\frac{e^{-3s}}{s}$$

$$= e^{-3s}\frac{2}{s^3} + 6e^{-3s}\frac{1}{s^2} + 9e^{-3s}\frac{1}{s}$$

EX: $u(t-\pi)\sin t$



$$u(t-\pi)\sin[\overbrace{(t-\pi)+\pi}^{t}]$$

$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$

$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$

Im part!

$$= u(t-\pi)[\cos(t-\pi)\underbrace{\sin\pi}_{0} + \sin(t-\pi)\underbrace{\cos\pi}_{-1}]$$

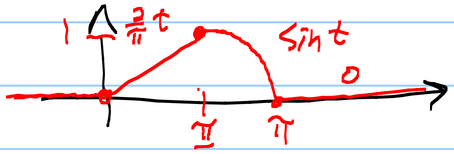
$$= -u(t-\pi)\underbrace{\sin(t-\pi)}$$

$f(t-\pi)=\sin(t-\pi)$

$f(t)=\sin t$

$$\mathcal{L} = -e^{-\pi s}\frac{1}{s^2+1}$$

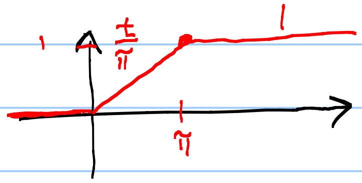
EX:



$$f(t) = [\overbrace{\underset{\uparrow \text{on at } 0}{u(t-0)} - \underset{\uparrow \text{off at } \frac{\pi}{2}}{u(t-\tfrac{\pi}{2})}}^{\text{switch}}]\left(\frac{2}{\pi}t\right) + [\underset{\uparrow \text{on at } \frac{\pi}{2}}{u(t-\tfrac{\pi}{2})} - \underset{\uparrow \text{off at } \pi}{u(t-\pi)}](\sin t) + \underset{\uparrow \text{on at } \pi}{u(t-\pi)}\cdot 0$$

(Never off)

EX: $y'' + y =$



$g(t)$ $\quad \begin{cases} y(0)=0 \\ y'(0)=0 \end{cases}$

$$g(t) = \left[u(t-0) - u(t-\pi)\right]\left(\frac{t}{\pi}\right) + u(t-\pi)\cdot 1$$

$$= \frac{1}{\pi}u(t-0)(t-0) - \frac{1}{\pi}u(t-\pi)\overbrace{\left[(t-\pi)+\underline{\underline{\pi}}\right]}^{t} + u(t-\pi) \quad \text{cancels}$$

$$= \frac{1}{\pi}u(t-0)(t-0) - \frac{1}{\pi}u(t-\pi)\underbrace{(t-\pi)}_{f(t-\pi)=(t-\pi)}$$

so $f(t) = t$

$$G(s) = \frac{1}{\pi}\underbrace{e^{-0s}}_{1}\frac{1}{s^2} - \frac{1}{\pi}e^{-\pi s}\frac{1}{s^2}$$

$$= \frac{1}{\pi}\left(1-e^{-\pi s}\right)\frac{1}{s^2}$$

$$\mathcal{L}\left[y''+y\right] = \mathcal{L}[g]$$

$$\left(s^2 Y - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0}\right) + Y = G(s)$$

$$Y = \underset{\substack{\uparrow \\ \text{transfer} \\ \text{fcn}}}{\frac{1}{s^2+1}}G(s) = \underbrace{\frac{1}{\pi}\frac{1}{s^2(s^2+1)}}_{H(s)} - \underbrace{\frac{1}{\pi}\frac{1}{s^2(s^2+1)}}_{H(s)}e^{-\pi s}$$

$$\underline{y = h(t) - u(t-\pi)h(t-\pi)}$$

$$H(s) = \frac{\overset{1/\pi}{A}}{s^2} + \frac{\overset{0}{B}}{s} + \frac{\overset{0}{C}s+\overset{1/\pi}{D}}{s^2+1} \underset{\substack{\uparrow \\ \text{do it}}}{=} \frac{1/\pi}{s^2} - \frac{1/\pi}{s^2+1}$$

So $h(t) = \frac{1}{\pi}\left(t - \sin t\right)$

$$y(t) = \frac{1}{\pi}\left(t-\sin t\right) - u(t-\pi)\frac{1}{\pi}\left((t-\pi) - \underbrace{\sin(t-\pi)}_{-\sin t}\right)$$

$$= \begin{cases} \frac{1}{\pi}\left(t - \sin t\right) & 0 \le t \le \pi \\ 1 - \frac{2}{\pi}\sin t & t > \pi \end{cases}$$