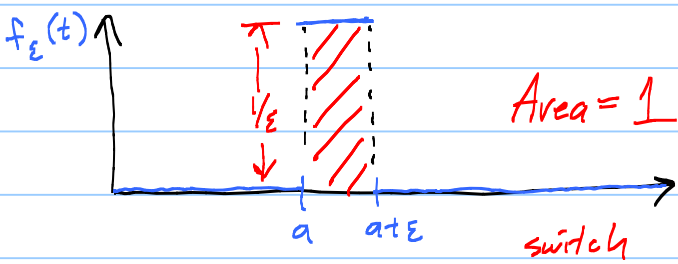
Lesson 21 on 6.4

18, 19, 20 due tonight
21, 22 next Wed.

WebEx tonight 8-9 pm

Impulse function (Dirac delta function)

$$\left[\begin{array}{ll} \text{s-shift rule} & \mathcal{L}[e^{ct}f(t)] = F(s-c) \\ \text{t-shift} & \mathcal{L}[u(t-c)f(t-c)] = e^{-cs}F(s) \end{array}\right.$$



$$f_\varepsilon(t) = \underbrace{\left[u(t-a) - u(t-(a+\varepsilon))\right]}_{\text{switch}} \underbrace{\left(\frac{1}{\varepsilon}\right)}_{\text{fcn}}$$

on at a off at $a+\varepsilon$

$$= \left(\frac{e^{-as}}{s} - \frac{e^{-(a+\varepsilon)s}}{s}\right)\left(\frac{1}{\varepsilon}\right)$$

$$= -\frac{1}{s}\underbrace{\left(\frac{e^{-(a+\varepsilon)s} - e^{-as}}{\varepsilon}\right)}_{DQ}$$

$G(a) = e^{-as}$ $\underbrace{\dfrac{G(a+\varepsilon) - G(a)}{\varepsilon}}_{DQ!}$

$G'(a) = -se^{-as}$

$DQ \longrightarrow G'(a) = -se^{-as}$

$$\lim_{\varepsilon \to 0} \mathcal{L}[f_\varepsilon(t)] = -\frac{1}{s}\left(-se^{-as}\right) = e^{-as}$$

Think: limit is perfect sledge hammer that hits at time a with unit impulse.

Notation: $\delta(t-a) \text{ "=" } \lim_{\varepsilon \to 0} f_\varepsilon(t)$

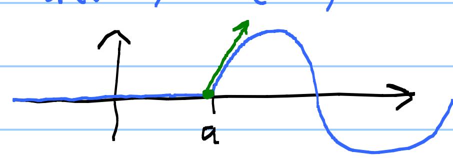
$$\mathcal{L}[\delta(t-a)] = e^{-as} \longleftarrow = \lim_{\varepsilon \to 0} \mathcal{L}[f_\varepsilon]$$

Problem: $y'' + y = f_\varepsilon(t)$ $\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$

$$(s^2+1)Y = F_\varepsilon(s)$$

$$Y = \frac{1}{s^2+1}F_\varepsilon(s) \xrightarrow[\varepsilon \to 0]{} \frac{e^{-as}}{s^2+1}$$

$$\left.\begin{array}{l}\text{Limiting } y \\ \text{as } \varepsilon \to 0\end{array}\right) = u(t-a)\sin(t-a)$$



Impulse = Δ(momentum)

$$F = ma = m\frac{dv}{dt}$$

$$\int_{t_1}^{t_2} F(t)\,dt = m\int_{t_1}^{t_2} \frac{dv}{dt}\,dt = \underbrace{m\left(v(t_2) - v(t_1)\right)}_{\Delta mv}$$

Key properties of $\delta(t-a) = \delta_a(t)$

1) $\delta(t-a) = 0$ if $t \neq a$.

2) $\int \delta(t-a)\,dt = 1$

3) If h is continuous, then

$$\int h(t)\underbrace{\delta(t-a)}_{\text{a distribution}}\,dt = h(a) \leftarrow \begin{array}{c}\text{Limit}\\ \varepsilon\to 0\\ f_\varepsilon\end{array}$$

4) $$\mathcal{L}[\delta(t-a)] = \int_0^\infty \underbrace{e^{-st}}_{h(t)}\,\delta(t-a)\,dt = e^{-as} \checkmark$$

EX: $y'' + 4y' + 13y = \delta(t-\pi)$ $\quad \begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$

$$\left(s^2 Y - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0}\right) + 4\left(sY - \underbrace{y(0)}_{0}\right) + 13Y = e^{-\pi s}$$

$$Y = \frac{1}{\underbrace{s^2+4s+13}_{\text{complex roots}}} e^{-\pi s}$$

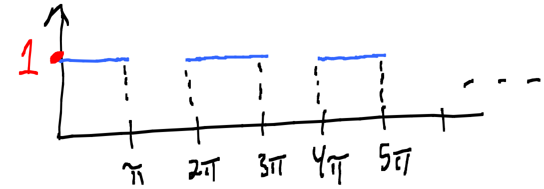
$$Y = \frac{1}{(s+\underset{\frac{1}{2}\cdot 4}{2})^2 + 9} e^{-\pi s} = \underbrace{\frac{1}{3}\cdot\frac{3}{(s+2)^2+3^2}}_{\mathcal{L}\left[\frac{1}{3}e^{-2t}\sin 3t\right]} e^{-\pi s}$$

So $$y = u(t-\pi)\,\frac{1}{3}\,e^{-2(t-\pi)}\sin 3(t-\pi)$$

Clean up: $e^{-2(t-\pi)} = e^{-2t+2\pi} = e^{2\pi}e^{-2t}$

$$\sin 3(t-\pi) = \sin(3t - 3\pi) = -\sin 3t$$

$$y = \begin{cases} 0 & t < \pi \\ -\frac{1}{3}e^{2\pi}e^{-2t}\sin 3t & t \geq \pi \end{cases}$$

EX: $y'' + y = g(t) = $  (Square wave)

$$g(t) = [u(t-0) - u(t-\pi)] + [u(t-2\pi) - u(t-3\pi)] + \cdots$$

$$G(s) = \left(\frac{1}{s} - \frac{e^{-\pi s}}{s}\right) + \left(\frac{e^{-2\pi s}}{s} - \frac{e^{-3\pi s}}{s}\right) + \cdots$$

$$= \frac{1}{s}\left(1 - e^{-\pi s} + e^{-2\pi s} - e^{-3\pi s} + \cdots\right)$$

Aha! $e^{-n\pi s} = \left(e^{-\pi s}\right)^n$

Geom Series $\quad 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r} \quad |r| < 1$

Hmmm: Replace r by $(-r)$:

$$1 - r + r^2 - r^3 + \cdots = \frac{1}{1-(-r)}$$

$$G(s) = \frac{1}{s} \frac{1}{1 + e^{-\pi s}} \qquad \left(\begin{array}{l} e^{-\pi s} < 1 \\ \text{if } s > 0 \end{array}\right)$$

Cool, but Σ more useful

Back to ODE:

$$(s^2+1)Y = \frac{1}{s}\left(1 - e^{-\pi s} + \cdots\right)$$

$$Y = \underbrace{\frac{1}{s(s^2+1)}}_{H(s)}\left(1 - e^{-\pi s} + \cdots\right)$$

$$H(s) = \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$h(t) = 1 - \cos t$$

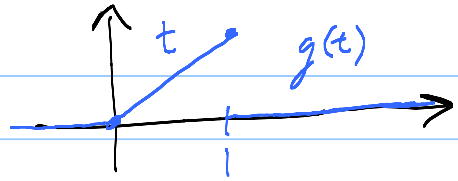
Fact: $\cos(t - (\text{EVEN})\pi) = \cos t$

$\cos(t - (\text{ODD})\pi) = -\cos t$

Get
$$y = \Big(\overbrace{1 - u(t-\pi) + u(t-2\pi) - u(t-3\pi) + \cdots}^{g(t)}\Big)$$
$$- (\cos t)\Big(\underbrace{1 + u(t-\pi) + u(t-2\pi) + u(t-3\pi) + \cdots}_{\text{staircase fcn}}\Big)$$

Resonance!

223: 25. $\quad y'' + y = \begin{cases} t & 0<t<1 \\ 0 & t>1 \end{cases} = g(t)$



$$g(t) = \left[u(t-0) - u(t-1)\right] t$$

$$= \underbrace{u(t-0)(t-0)}_{\substack{\mathcal{L}\text{ only sees} \\ 1\cdot t}} - \underbrace{u(t-1)\underbrace{\left[(t-1)+1\right]}_{t}}_{u(t-1)\underbrace{(t-1)}_{\substack{f(t-1)=t-1 \\ f(t)=t}} - u(t-1)}$$

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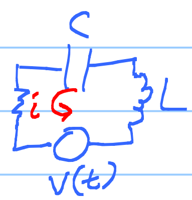

$$L\frac{di}{dt} + Ri + \frac{1}{C}\underbrace{\int_0^t i(\tau)\,d\tau}_{Q} = v(t)$$

Assume $i(0)=0$

Take $\mathcal{L}$: $\quad L\left(sI - \underbrace{i(0)}_{0}\right) + RI + \frac{1}{C}\left(\frac{1}{s}I\right) = \bar{V}(s)$