Lesson 22 on 6.5 Convolution

21, 22 due Wed
WebEx Weds 8-9 pm

EX: $y'' + 4y = \sin 2t \quad \begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$

$$\Big(s^2 Y - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0}\Big) + 4Y = \frac{2}{s^2+2^2}$$

$$(s^2+4)Y = \frac{2}{s^2+4}$$

$$Y = \frac{2}{(s^2+4)^2}$$

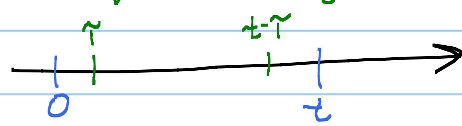
$$\left[\ \sqrt{ab} = \pm\sqrt{a}\sqrt{b} \qquad a, b \text{ complex!} \qquad e^{a+b} = e^a e^b \qquad e^0 = 1\right.$$

No! $\cancel{\mathcal{L}[\sin t \cdot \sin t] = \frac{2}{s^2+4} \cdot \frac{2}{s^2+4}}$

$$\frac{1}{s} = \mathcal{L}[1 \cdot 1] \stackrel{?}{=} \frac{1}{s} \cdot \frac{1}{s} \leftarrow \text{No!}$$

True thing: $\mathcal{L}[f * g] = F(s)\, G(s)$

where $(f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$



$$\left[\ \text{Discrete version: } \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n \right.$$

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

EX: Find $\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right]$.

$$\mathcal{L}[\underset{\sin t}{f} * \underset{\sin t}{g}] = \underbrace{\frac{1}{s^2+1}}_{F(s)} \cdot \underbrace{\frac{1}{s^2+1}}_{G(s)}$$

Ans: $(f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$

$$= \int_0^t \sin \underbrace{\tau}_{\alpha}\, \sin \underbrace{(t-\tau)}_{\beta}\, d\tau$$

Trig: $\sin\alpha \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$

$$= \int_0^t \frac{1}{2}\Big[\cos\big(\tau - (t-\tau)\big) - \cos\big(\tau + (t-\tau)\big)\Big]\, d\tau$$

$$= \frac{1}{2}\int_{\tau=0}^{t} \cos\underbrace{(2\tau - t)}_{u} - \cos t \, d\tau$$

$u = 2\tau - t$ When $\tau = 0$, $u = 2\cdot 0 - t = -t$

$du = 2\,d\tau$ $\quad\tau = t$, $u = 2t - t = t$

$\boxed{d\tau = \frac{1}{2}du}$

$$= \frac{1}{2}\int_{u=-t}^{t} \cos u \underbrace{\left(\tfrac{1}{2}du\right)}_{d\tau} - \frac{1}{2}\cos t \underbrace{\int_0^t 1\, d\tau}_{t}$$

$$= \frac{1}{4}\Big[\sin u\Big]_{-t}^{t} - \frac{1}{2}t\cos t$$

$$= \frac{1}{4}\Big(\sin t - \underbrace{\sin(-t)}_{-\sin t}\Big) - \frac{1}{2}t\cos t$$

$$= \frac{1}{2}\sin t - \frac{1}{2}t\cos t \longleftarrow \text{Ans.}$$

In 6.6: $\mathcal{L}[t f(t)] = -F'(s)$

$$\mathcal{L}[t\cos t] = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]$$

<u>Fact</u>: $f * g = g * f$

<u>Why</u>: $\mathcal{L}[f*g] = F(s)G(s) = G(s)F(s) = \mathcal{L}[g*f]$

✓ Uniqueness of L. Transf!

<u>EX</u>: $y'' + y = r(t)$ $\quad\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$

$$(s^2+1)Y = R(s)$$

$$Y = \frac{1}{s^2+1}\cdot R(s)$$

So
$$y = (\sin t) * r(t)$$
$$= r(t) * \sin t$$
$$= \int_0^t r(\tau)\underbrace{\sin(t-\tau)}_{\text{Kernel of integral operator}}d\tau$$

Why: $\mathcal{L}[f*g] = F(s)G(s)$

$$F(s)G(s) = \left(\int_0^\infty e^{-su} f(u)\,du\right)\left(\int_0^\infty e^{-sv} g(v)\,dv\right)$$

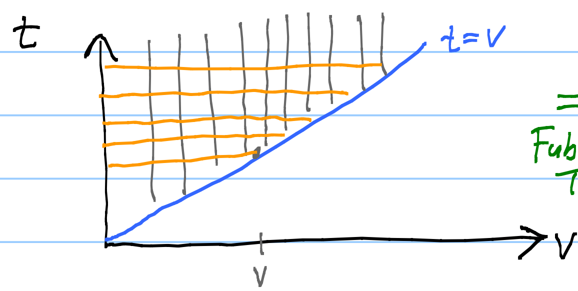
$$= \int_{v=0}^\infty \left(\int_{u=0}^\infty e^{-s\underbrace{(u+v)}_{t}} f(u)\,g(v)\,du\right) dv$$

Let $t = u+v$ then $u = t-v$

when $u=0$, $t = 0+v = v$

As $u\to\infty$, $t\to\infty$

$$= \int_{v=0}^\infty g(v)\left(\int_{t=v}^\infty e^{-st} f(t-v)\,dt\right) dv$$



$$\underset{\text{Fubinni's Thm}}{=} \int_{t=0}^{\infty} e^{-st} \underbrace{\int_{v=0}^{t} g(v) f(t-v)\,dv}_{(g*f)(t)\ \checkmark}\, dt$$

Integral Eqns: Voltera integral eqns of the second kind.

EX: $y(t) - \int_0^t y(\tau)\sin(t-\tau)\,d\tau = t$

Hit with $\mathcal{L}$: $Y - Y\cdot\frac{1}{s^2+1} = \frac{1}{s^2}$

$$\left(\frac{s^2+1}{s^2+1} - \frac{1}{s^2+1}\right)Y = \frac{1}{s^2}$$

$$\frac{(s^2+1)-1}{s^2+1} = \frac{s^2}{s^2+1}$$

$$Y = \frac{1}{s^2}\cdot\frac{(s^2+1)}{s^2}$$

$$= \frac{1}{s^4}(s^2+1)$$

$$= \frac{1}{s^2} + \frac{1}{s^4}$$

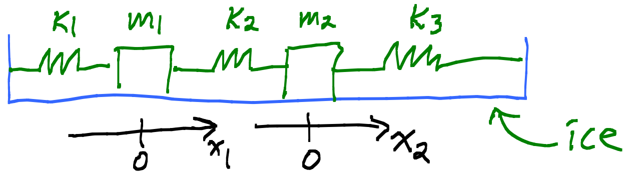
$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[t^{n-1}] = \frac{(n-1)!}{s^n}$$

$$\mathcal{L}\left[\frac{1}{(n-1)!}t^{n-1}\right] = \frac{1}{s^n}$$

So $y = t + \frac{1}{3!}t^3$

Prob:



$$\begin{cases} m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 \ddot{x}_2 = -k_3 x_2 - k_2(x_2 - x_1) \end{cases}$$

Take all const $= 1$

$$\begin{cases} \ddot{x}_1 = -2x_1 + x_2 & (A) \\ \ddot{x}_2 = x_1 - 2x_2 & (B) \end{cases}$$

Undergrad method: Solve (B) for x_1: $\boxed{x_1 = \ddot{x}_2 + 2x_2}$

Plug into (A): $\big(\underbrace{\ddot{x}_2 + 2x_2}_{x_1}\big)'' = -2\big(\underbrace{\ddot{x}_2 + 2x_2}_{x_1}\big) + x_2$

$$x_2^{(4)} + 4\ddot{x}_2 + 3x_2 = 0$$

$$r^4 + 4r^2 + 3 = 0$$

$$(r^2+1)(r^2+3) = 0 \qquad r = \pm i, \pm\sqrt{3}\, i$$

$$x_2 = c_1 \sin t + c_2 \cos t + c_3 \sin\sqrt{3}\,t + c_4 \cos\sqrt{3}\,t$$

$$x_1 = \ddot{x}_2 + 2x_2 = c_1 \sin t + c_2 \cos t - c_3 \sin\sqrt{3}\,t - c_4 \cos\sqrt{3}\,t$$

Modes of oscillation: Take one $c = 1$, rest $= 0$

$x_1 = \sin t$	$\cos t$	$-\sin\sqrt{3}\,t$	$-\cos\sqrt{3}\,t$
$x_2 = \sin t$	$\cos t$	$\sin\sqrt{3}\,t$	$\cos\sqrt{3}\,t$

Sol^n with $x_1(0) = 1 \quad x_2(0) = 0$

$\dot{x}_1(0) = 0 \quad \dot{x}_2(0) = 0$

$$x_1(t) = \tfrac{1}{2}\left(\cos\sqrt{3}\,t + \cos t\right)$$
$$= \left(\cos\tfrac{\sqrt{3}+1}{2}t\right)\left(\cos\tfrac{\sqrt{3}-1}{2}t\right)$$

$$\cos\alpha + \cos\beta = 2\cos\tfrac{\alpha+\beta}{2}\cos\tfrac{\alpha-\beta}{2}$$

$$x_2(t) = \tfrac{1}{2}\left(-\cos\sqrt{3}\,t + \cos t\right)$$
$$= \left(\sin\tfrac{\sqrt{3}+1}{2}t\right)\left(\sin\tfrac{\sqrt{3}-1}{2}t\right)$$

$$-\cos\alpha + \cos\beta = 2\sin\tfrac{\alpha+\beta}{2}\sin\tfrac{\alpha-\beta}{2}$$

Think: $\frac{\sqrt{3}+1}{2}$ fast

$\frac{\sqrt{3}-1}{2}$ slow