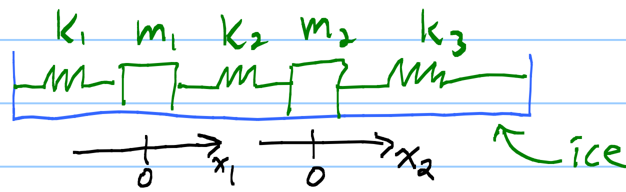
Lesson 24 on 6.7 and 12.2

21, 22 due tonight
WebEx 8-9 pm
23, 24, 25 due next

Prob:



$$\begin{cases} m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1) \\ m_2\ddot{x}_2 = -k_3x_2 - k_2(x_2-x_1) \end{cases}$$

Take all const $= 1$

$$\begin{cases} \ddot{x}_1 = -2x_1 + x_2 & (A) \\ \ddot{x}_2 = x_1 - 2x_2 & (B) \end{cases}$$

Undergrad method: Solve (B) for x_1: $\boxed{x_1 = \ddot{x}_2 + 2x_2}$

Plug into (A): $\underbrace{(\ddot{x}_2 + 2x_2)}_{x_1}'' = -2\underbrace{(\ddot{x}_2 + 2x_2)}_{x_1} + x_2$

$$x_2^{(4)} + 4\ddot{x}_2 + 3x_2 = 0$$

$$r^4 + 4r^2 + 3 = 0$$

$$(r^2+1)(r^2+3) = 0 \qquad r = \pm i, \pm\sqrt{3}\,i$$

$$x_2 = c_1 \sin t + c_2 \cos t + c_3 \sin\sqrt{3}\,t + c_4 \cos\sqrt{3}\,t$$

$$x_1 = \ddot{x}_2 + 2x_2 = c_1 \sin t + c_2 \cos t - \underline{c_3} \sin\sqrt{3}t - \underline{c_4} \cos\sqrt{3}t$$

Modes of oscillation: Take one $c=1$, rest $=0$

$x_1 = \sin t$	$\cos t$	$-\sin\sqrt{3}t$	$-\cos\sqrt{3}t$
$x_2 = \sin t$	$\cos t$	$\sin\sqrt{3}t$	$\cos\sqrt{3}t$

Sol^n with $x_1(0) = 1 \quad x_2(0) = 0$

$\dot{x}_1(0) = 0 \quad \dot{x}_2(0) = 0$

$$x_1(t) = \tfrac{1}{2}\left(\cos\sqrt{3}t + \cos t\right) \leftarrow$$
$$= \left(\cos\tfrac{\sqrt{3}+1}{2}t\right)\left(\cos\tfrac{\sqrt{3}-1}{2}t\right)$$

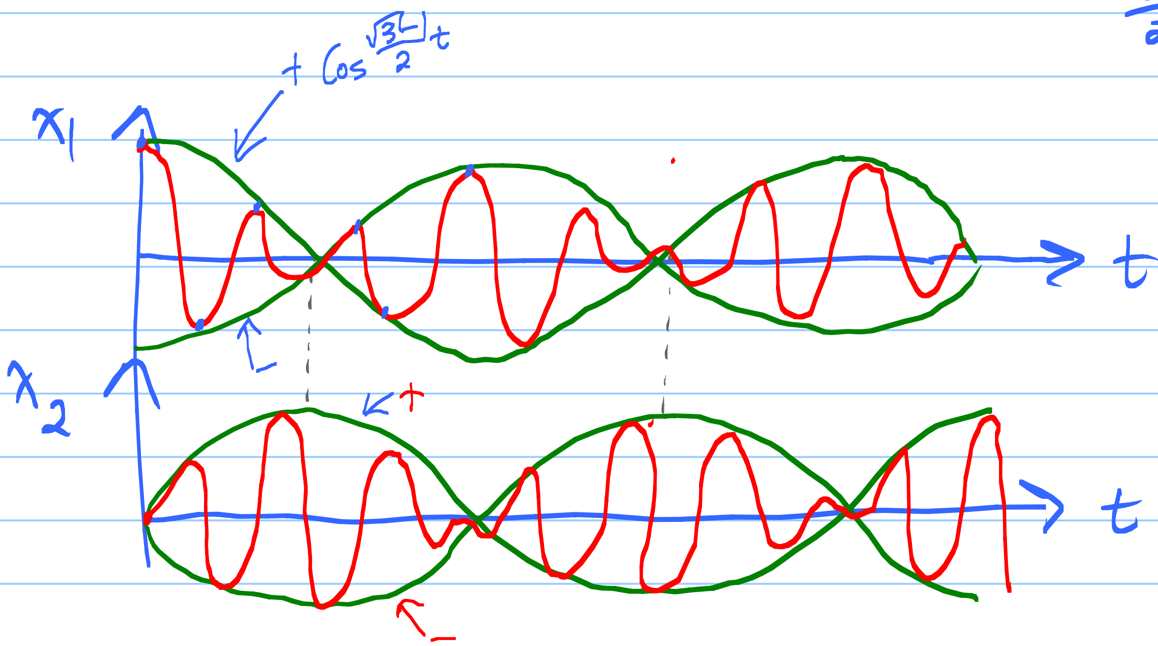
$$\cos\alpha + \cos\beta = 2\cos\tfrac{\alpha+\beta}{2}\cos\tfrac{\alpha-\beta}{2}$$

$$x_2(t) = \tfrac{1}{2}\left(-\cos\sqrt{3}t + \cos t\right)$$
$$= \left(\sin\tfrac{\sqrt{3}+1}{2}t\right)\underline{\left(\sin\tfrac{\sqrt{3}-1}{2}t\right)}$$

$$-\cos\alpha + \cos\beta = 2\sin\tfrac{\alpha+\beta}{2}\sin\tfrac{\alpha-}{2}$$

Think: $\frac{\sqrt{3}+1}{2}$ fast

$\frac{\sqrt{3}-1}{2}$ slow



Use $\mathcal{L}$ to solve $\begin{cases} \ddot{x}_1 = -2x_1 + x_2 \\ \ddot{x}_2 = x_1 - 2x_2 \end{cases} \leftarrow$ Apply $\mathcal{L}$

I.C. $x_1(0) = 1 \quad x_2(0) = 0$

$\dot{x}_1(0) = 0 \quad \dot{x}_2(0) = 0$

$$\begin{cases} s^2 X_1 - s\underbrace{x_1(0)}_{1} - \underbrace{\dot{x}_1(0)}_{0} = -2X_1 + X_2 \\ s^2 X_2 - s\underbrace{x_2(0)}_{0} - \underbrace{\dot{x}_2(0)}_{0} = X_1 - 2X_2 \end{cases}$$

$$\begin{cases} -s = (-2-s^2)X_1 + X_2 \\ 0 = X_1 + (-2-s^2)X_2 \end{cases}$$

$$\begin{pmatrix} -s \\ 0 \end{pmatrix} = \begin{bmatrix} -2-s^2 & 1 \\ 1 & -2-s^2 \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

Cramer's:

$$X_1 = \frac{\det\begin{bmatrix} -s & 1 \\ 0 & -2-s^2 \end{bmatrix}}{\det\begin{bmatrix} -2-s^2 & 1 \\ 1 & -2-s^2 \end{bmatrix}} = \frac{s(s^2+2)}{(s^2+2)^2-1}$$

$$= \frac{s^3+2s}{s^4+4s^2+3}$$

$$= \frac{s^3+2s}{(s^2+1)(s^2+3)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+3}$$

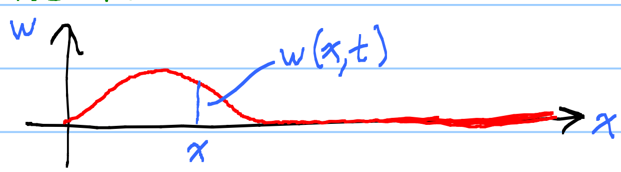
Get x_1 = linear combo of
$\sin t$, $\cos t$, $\sin\sqrt{3}\,t$, $\cos\sqrt{3}\,t$

Next $$X_2 = \frac{\det\begin{bmatrix} -2-s^2 & -s \\ 1 & 0 \end{bmatrix}}{s^4+4s^2+3} = \frac{s}{(s^2+1)(s^2+3)}$$

has same partial fraction decomposition

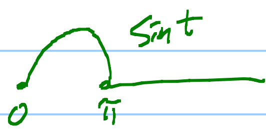
12.12 Using $\mathcal{L}$ to solve PDE:

Prob: Whip a rope.



Wave eqn: $$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$ (c = speed of propogation)

Whip left end: $w(0,t) =$



$= f(t)$

$$= [u(t-0) - u(t-\pi)]\sin t$$

Energy considerations: $\lim\limits_{x\to\infty} w(x,t) = 0$ ← t held fixed

Initial conditions $\begin{cases} w(x,0)=0 \\ \frac{\partial w}{\partial t}(x,0)=0 \end{cases}$ "dead rope"

Hit PDE with $\mathcal{L}$ in t-variable.

$$\mathcal{L}\left[w(x,t)\right] = \int_0^\infty e^{-st} w(x,t)\,dt \overset{\text{def}}{=} \mathbb{W}(x,s)$$

$\mathcal{L}$: $$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

$$s^2\mathbb{W} - s\underbrace{w(x,0)}_{0} - \underbrace{\frac{\partial w}{\partial t}(x,0)}_{0} = c^2 \int_0^\infty e^{-st} \frac{\partial^2}{\partial x^2} w(x,t)\,dt$$

$$= c^2 \frac{\partial^2}{\partial x^2}\left(\int_0^\infty e^{-st} w(x,t)\,dt\right)$$

$$= c^2 \frac{\partial^2}{\partial x^2} \mathbb{W}(x,s)$$

$$s^2\mathbb{W} = c^2 \frac{\partial^2 \mathbb{W}}{\partial x^2}$$

$$\frac{\partial^2 \mathbb{W}}{\partial x^2} - \frac{s^2}{c^2}\mathbb{W} = 0$$ ← 2nd order linear ODE in x-var. (s is a parameter)

$$\left[y'' - k^2 y = 0, \quad y = c_1 e^{kx} + c_2 e^{-kx}\right]$$

$$\mathbb{W} = c_1(s)\, e^{sx/c} + c_2(s)\, e^{-sx/c}$$

Note: Laplace transforms → 0 as $s\to\infty$

Oops! $\mathcal{L}[\delta(t)] = 1$

Energy considerations: $\mathbb{W}(x,s) = \int_0^\infty e^{-st} \underbrace{w(x,t)}_{\to 0 \text{ as } x\to\infty}\,dt$

See $\mathbb{W}(x,s) \to 0$ as $x\to\infty$.

Aha! $c_1(s)$ must be zero for each s.

Get $\boxed{\overline{W}(x,s) = c_2(s)\, e^{-sx/c}}$

What is $c_2(s)$? $\quad \overline{W}(0,s) = c_2(s)$

$$= \int_0^\infty e^{-st} \underbrace{w(0,t)}_{\text{one hump of sine}} dt$$

$$= F(s)$$

So $\overline{W}(x,s) = F(s)\, e^{-sx/c}$

Undo $\mathcal{L}$: $\quad w(x,t) = \mathcal{L}^{-1}\left[e^{-sx/c} F(s)\right]$

$$= u\left(t-\tfrac{x}{c}\right) f\left(t-\tfrac{x}{c}\right)$$

one hump of sine