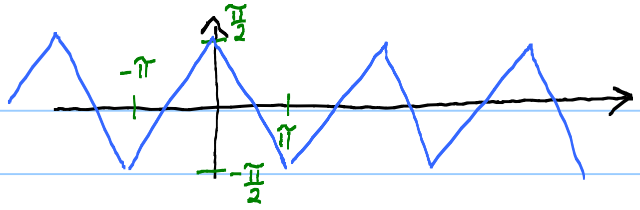
Lesson 27 on 11.3, 11.4

HWK 8: 23,24,25 due tonight
HWK 9: 26,27,28 due Wed. next

11.3

Spring mass

$$\underset{m=1}{\underset{\uparrow}{y''}} + \underset{\text{c due to friction}}{\underset{\uparrow}{0.05}}y' + \underset{k}{\underset{\uparrow}{25}}y = \underset{\text{forcing fcn}}{\underset{\uparrow}{F(x)}} =$$

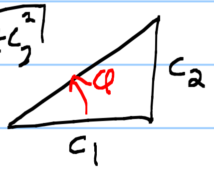


Frictionless homog prob: $y'' + 25y = 0$

$$r^2 + 25 = 0$$
$$r = \pm 5i$$

$$y = c_1 \cos 5x + c_2 \sin 5x$$
$$= \sqrt{c_1^2+c_2^2}\left(\underbrace{\frac{c_1}{\sqrt{c_1^2+c_2^2}}}_{\cos\varphi} \cos 5x + \underbrace{\frac{c_2}{\sqrt{c_1^2+c_2^2}}}_{\sin\varphi} \sin 5x\right)$$
$$= A\cos(5x - \varphi)$$

$A = \sqrt{c_1^2+c_2^2}$



Homog Prob with friction: $y'' + .05y' + 25y = 0$

$$r^2 + .05r + 25 = 0$$

$$r = -\frac{.05}{2} \pm \frac{\sqrt{(.05)^2 - 100}}{2} = \left(\text{negative small}\right) \pm \left(\text{close to } 5\right)i$$
$$= -a \pm bi$$

$$y = c_1 \underset{\uparrow}{e^{-ax}} \cos bx + c_2 \underset{\uparrow}{e^{-ax}} \sin bx$$

friction causes sol^ns to die away

This homog. is sol^n is transient.

Back to big prob: Need particular sol^n.

Plan: Expand F in F.S.

$$F(x) = \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right)$$
$$= \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}\cos(2n+1)x$$

Fun fact: $F(0) = \frac{\pi}{2} = \frac{4}{\pi}\underbrace{\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)}_{=\frac{\pi^2}{8}}$

Plan: Get part. sol^n y_{P_n} for RHS

$$\frac{4}{\pi n^2}\cos nx \leftarrow n \text{ odd}$$

Then $\underline{\underline{y_P}} = y_{P_1} + y_{P_3} + y_{P_5} + \cdots$

Prob: $y'' + .05y' + 25y = \frac{4}{\pi n^2}\cos nx$

Undet Coeff:

$$\begin{array}{rll} & y_{P_n} = A_n\cos nx + B_n\sin nx & \times\, 25 \\ & y'_{P_n} = -nA_n\sin nx + nB_n\cos nx & \times\, .05 \\ + & \underline{y''_{P_n} = -n^2A_n\cos nx - n^2B_n\sin nx} & \times\, 1 \end{array}$$

$$\Big(\underbrace{-n^2A_n + .05nB_n + 25A_n}_{\frac{4}{\pi n^2}}\Big)\cos nx + \Big(\underbrace{-n^2B_n - .05nA_n + 25B_n}_{0}\Big)\sin nx \overset{\text{want}}{=}$$

$$\frac{4}{\pi n^2}\cos nx + 0\sin nx$$

Cramer's: $\begin{cases} (25-n^2)A_n + .05nB_n = \frac{4}{\pi n^2} \\ -.05\,nA_n + (25-n^2)B_n = 0 \end{cases}$

$$A_n = \frac{4(25-n^2)}{\pi n^2\left[\underline{(25-n^2)}^2 + (.05n)^2\right]}$$

$$B_n = \frac{.2}{\pi n\left[\underline{(25-n^2)}^2 + (.05n)^2\right]}$$

Something big happens when $n=5$!

$$y_{P_n} = A_n\cos nx + B_n\sin nx$$

$$= \underbrace{\sqrt{A_n^2 + B_n^2}}_{\text{Amplitude}}\cos(nx - \varphi_n)$$

$$\text{Amplitude} = \frac{4}{n^2\pi^2\sqrt{(25-n^2)^2 + (.05n)^2}}$$

Gigantic when $n=5$. Small for rest.

Think: $n=5$ close to natural freq of spring-mass. Sympathetic vibrations.

11.4 Orthogonal expansions

$L^2[-\pi,\pi] = f$ on $[-\pi,\pi]$ such that $\int_{-\pi}^{\pi} f^2\,dx < \infty$

Inner product: $(\varphi,\psi) = \int_{-\pi}^{\pi} \varphi(x)\psi(x)\,dx$

Norm $\|\varphi\| = \sqrt{(\varphi,\varphi)} = \sqrt{\int_{-\pi}^{\pi} \varphi^2\,dx}$

Prob: Expand f in terms of $\perp$ φ_n

Orthonormal $(\varphi_n,\varphi_m) = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$

(Replace φ_n by $\dfrac{\varphi_n}{\|\varphi_n\|}$ to get)

Want $f = \sum_1^\infty c_n\varphi_n$

Step 1: Mult. by φ_m: $f\varphi_m = \sum_1^\infty c_n\varphi_n\varphi_m$

Step 2: $\int_{-\pi}^{\pi}$: $\int_{-\pi}^{\pi} f\varphi_m\,dx = \sum_1^\infty c_n \underbrace{\int_{-\pi}^{\pi}\varphi_n\varphi_m\,dx}_{=0} = c_m$

except $n=m$ term.

$$\boxed{c_m = \int_{-\pi}^{\pi} f(x)\varphi_m(x)\,dx}$$

Fact: Among all linear combos $\sum_1^N A_n\varphi_n$, the choice $A_n = c_n$, $n=1,\dots,N$ makes $\|f - \sum_1^N A_n\varphi_n\|$ an absolute min.

Why: $\overset{E}{E^*} = \|f - \sum_1^N \overset{A_n}{c_n}\varphi_n\|^2$

$$= \int_{-\pi}^{\pi}\Big(f - \sum_1^N \overset{A_n}{c_n}\varphi_n\Big)\Big(f - \sum_1^N \overset{A_m}{c_m}\varphi_m\Big)dx$$

$$= \int_{-\pi}^{\pi} f^2\,dx - \sum_1^N \overset{A_n}{c_n}\underbrace{\int_{-\pi}^{\pi}\varphi_n f\,dx}_{c_n} - \sum_1^N \overset{A_m}{c_m}\underbrace{\int_{-\pi}^{\pi}\varphi_m f\,dx}_{c_m} + \sum_{n=1}^N\sum_{m=1}^N \overset{A_n}{c_n}\overset{A_m}{c_m}\underbrace{\int_{-\pi}^{\pi}\varphi_n\varphi_m\,dx}_{\delta_{nm} = \begin{cases}1 & n=m\\ 0 & n\neq m\end{cases}}$$

$$= \int_{-\pi}^{\pi} f^2 \, dx - \sum_{1}^{N} c_n^2 - \sum_{1}^{N} c_m^2 + \sum_{n=1}^{N} c_n^2$$

$$= \int_{-\pi}^{\pi} f^2 \, dx - \sum_{1}^{N} c_n^2$$

$0 \leq$ → this

Bessel's Ineq: $\displaystyle\sum_{n=1}^{\infty} c_n^2 \leq \int_{-\pi}^{\pi} f^2 \, dx$

$$\left[E - E^* = \sum_{n=1}^{N} (A_n - c_n)^2 \leftarrow \text{sum of squares} \right]$$

≥ 0 ← only $= 0$ when $c_n = A_n$ for all n

<u>Word</u>: $\displaystyle a_0 + \sum_{n=1}^{N} \left(a_n \cos nx + b_n \sin nx \right)$ ← Trigonometric polys

<u>Fact</u>: N-th partial sum of a F.S. is <u>best</u> L^2-approx of f among N-th deg. Trig polys.

Parseval's Identity: $\displaystyle 2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2 \, dx$

Remark: Parseval's Ident. holds when φ_n form <u>complete</u> orthonormal basis for L^2.