Lesson 28 on 11.5 Sturm-Liouville Problems

26, 27, 28 due Wed.
WebEx Office Hr. Tues 8-9 pm Eastern Time

Ex: $y'' - \lambda y = 0$ ODE

B.C. $y(0) = 0,\ y(\pi) = 0$

Eigenvalues λ are special values that allow non-zero sol'ns to ODE + BC. Non-zero sol'ns are called eigenfunctions.

Big fact: The e-fcns are $\perp$.

Why: Suppose y_1 and y_2 are two e-fcns for $\lambda_1 \neq \lambda_2$.

$$\lambda_1 (y_1, y_2) = \int_0^{\pi} \underbrace{\lambda_1 y_1}_{= y_1''} y_2 \, dx = \int_0^{\pi} \underset{u}{y_2} \, \underbrace{y_1'' \, dx}_{dv}$$

$du = y_2' dx \quad \Big| \quad v = y_1'$

$$= uv \Big|_0^{\pi} - \int_0^{\pi} v \, du$$

$$= \underbrace{y_2 y_1' \Big|_0^{\pi}}_{=0 \text{ by B.C.}} - \int_0^{\pi} \underset{v}{y_1'} \, \underbrace{y_2' \, dx}_{du}$$

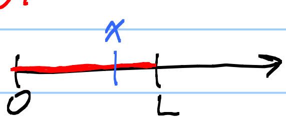
equal!

Aha! Repeat: $\lambda_2 (y_1, y_2) = \lambda_2 (y_2, y_1) = -\int_0^{\pi} y_2' y_1' \, dx$

So $\lambda_1 (y_1, y_2) = \lambda_2 (y_1, y_2)$

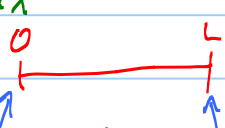
$$\underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \underbrace{(y_1, y_2)}_{\text{must } = 0.} = 0$$

Heat problem: Hot wire



$u(x,t)$ = temp of wire at x at time t.

Heat eqn: $\dfrac{\partial^2 u}{\partial x^2} = c \dfrac{\partial u}{\partial t}$



let radiate heat

$\dfrac{\partial u}{\partial x}(0,t) = -K u(0,t)$

ice this end

$u(L,t) = 0$

Take $c=1$, $k=1$. Try $u(x,t)=X(x)T(t)$

PDE: $X''(x)T(t) = X(x)T'(t)$

$$\boxed{\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = \lambda}$$

$X''(x) - \lambda X(x) = 0$ | $T'(t) - \lambda T(t) = 0$

$T(t) = ce^{\lambda t}$

B.C. 1) $u(L,t) = 0$

$$\underset{\uparrow}{X(L)}\,\underbrace{T(t)}_{} = 0$$

don't want $T(t) \equiv 0$.

Must $X(L) = 0$

2) $\frac{\partial u}{\partial x}(0,t) + u(0,t) = 0$ for all t

$X'(0)T(t) + X(0)T(t) = 0$

$$[\underbrace{X'(0) + X(0)}_{\text{must } = 0}]\,T(t) = 0$$

New Sturm-Liouville Prob: $X'' - \lambda X = 0$

B.C. $\begin{cases} X(L) = 0 \\ X'(0) + X(0) = 0 \end{cases}$

Sol'ns:
1) $X(x) = c_1 x + c_2$ if $\lambda = 0$
2) $X(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$ if $\lambda = \mu^2 > 0$
3) $X(x) = c_1 \cos \mu x + c_2 \sin \mu x$ if $\lambda = -\mu^2 < 0$

Look for non-zero sol'ns satisfying BC

General Sturm-Liouville Prob:

ODE: $[p(x)y]' + (q(x) + \lambda r(x))y = 0$

B.C.: $\begin{cases} k_1 y(a) + k_2 y'(a) = 0 & k_1, k_2 \text{ not both zero} \\ \ell_1 y(b) + \ell_2 y'(b) = 0 & \ell_1, \ell_2 \text{ not both zero} \end{cases}$

p, p', q, r continuous on $[a,b]$ and $r(x) > 0$ on (a,b).

Facts: e-vals λ must be real.

e-fcns are $\perp$ on (a,b) with respect to a weighted inner product

$$(y_1, y_2)_r = \int_a^b y_1(x)\, y_2(x)\, \underbrace{r(x)}_{\text{weight fcn}}\, dx$$

Remark: $y'' + \lambda y = 0$ $\quad p \equiv 1,\ q \equiv 0,\ r \equiv 1$

$y(0) = 0,\ y(\pi) = 0$ $\quad k_1 = 1,\ k_2 = 0$

$\ell_1 = 1,\ \ell_2 = 0$

$a = 0,\ b = \pi$

EX: p. 503: 13

$$(*) \quad y'' + 8y' + (16 + \lambda) y = 0 \quad \begin{cases} y(0) = 0 \\ y(\pi) = 0 \end{cases}$$

S-L: $[py']' + (q + \lambda r) y = 0$

$$p y'' + p' y' + (q + \lambda r) y = 0$$

Mult (*) by p: $p y'' + 8p y' + (16p + \lambda p) y = 0$

Aha! Need $\boxed{p' = 8p}$. So $\boxed{p = e^{8x}}$

$$(*): \quad [\underbrace{e^{8x}}_{p} y']' + (\underbrace{16 e^{8x}}_{q} + \lambda \underbrace{e^{8x}}_{r(x)}) y = 0$$

e-fcns $\perp$: $\displaystyle\int_0^{\pi} y_1(x)\, y_2(x)\, e^{8x}\, dx$

Warning: When looking for e-vals and non-zero e-fcns, easier to use ODE in original form rather than standard form.

Legendre Eqns: $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ on $(-1,1)$.

$$[\underbrace{(1-x^2)}_{P}y']' + \underset{q=0}{\uparrow}\ \lambda\ \underset{r=1}{\uparrow} y = 0$$

$\lambda = n(n+1)$ e-vals

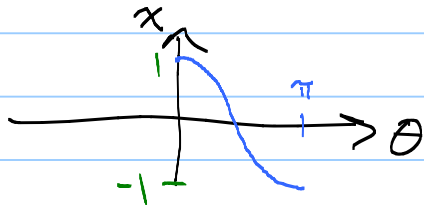
Get Legendre Polynomials $P_n(x)$.

They are $\perp$ on $(-1,1)$.

$$\int_{-1}^{1} P_n(x)P_m(x)\cdot 1\, dx = 0 \text{ if } n \neq m.$$

HWK: Show $P_n(\cos\theta)$ are $\perp$ with respect to weight fcn $\sin x$ on $(0,\pi)$.

Think: Change of variables: $x = \cos\theta$



Hmmm. $$\int_0^{\pi} P_n(\underbrace{\cos\theta}_{x})\, P_m(\underbrace{\cos\theta}_{x})\, \sin\theta\, d\theta$$

What is dx? When $\theta = 0$: $x = ?$

$\theta = \pi$: $x = ?$