Lesson 29 on 11.6 Generalized Fourier Series

HWK 9: 26, 27, 28 due Wed.
WebEx Off. Hr. Tues. 8-9 pm

Legendre Eqn: $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

$$y'' + Py' + Qy = 0 \leftarrow \text{Standard Form}$$

$P(x) = \frac{-2x}{1-x^2}$ $\quad Q(x) = \frac{n(n+1)}{1-x^2}$ $\quad$ Ouch! at $x = \pm 1$

"Singular Sturm Liouville Prob" (See Theorem 1 in 11.5)

Legendre Polys: p. 178.

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m}$$

where $M = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$

$P_0(x) = 1$ $\qquad P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$ $\qquad P_3(x) = \frac{1}{2}(5x^3-3x)$

$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$ $\qquad P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$

Facts: 1) $\text{Span}(P_0, P_1, \ldots, P_n) = \text{Span}(1, x, x^2, \ldots, x^n)$
$= \text{all polys of deg} \le n$

2) P_n is even for n even
P_n is odd for n odd

3) The P_n are $\perp$ on $(-1,1)$ $\quad [r(x) \equiv 1]$

4) The $L^2[-1,1]$ norm of P_n is $\sqrt{\frac{2}{2n+1}}$

$$\|P_n\| = \sqrt{(P_n, P_n)} = \sqrt{\int_{-1}^{1} P_n^2\, dx} = \sqrt{\frac{2}{2n+1}}$$

5) The P_n form a complete orthogonal basis for $L^2[-1,1]$, meaning that every f in $L^2[-1,1]$ can be expanded

$$f = \sum_{n=0}^{\infty} c_n P_n$$

$$\int_{-1}^{1} f(x) P_n(x)\, dx = c_n \int_{-1}^{1} P_n^2\, dx = \frac{2}{2n+1} c_n$$

$$\boxed{c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx}$$

6) Bessel's Ineq: $\displaystyle\sum_{n=0}^{\infty} \frac{2n+1}{2} |c_n|^2 \le \|f\|^2$

7) Parseval's Identity: $\displaystyle\sum_{n=0}^{\infty} \frac{2n+1}{2} |c_n|^2 \underset{\text{by completeness}}{=} \|f\|^2$

Use these facts to do p.509: 1, 3, 5

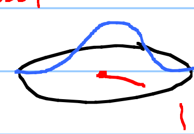
Ex: Expand x^4 in Leg. Polys.

Fact 1: $x^4 = c_0 P_0 + c_1 P_1 + \cdots + c_4 P_4$

c's given by $\perp$ property:

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} \underbrace{\underset{\text{even}}{x^4}\; \underset{\text{odd if } n \text{ odd}}{\underbrace{P_n(x)}}}_{\text{odd},\ n \text{ odd}}\, dx = 0, \ n \text{ odd}$$

Vibrating circular membrane:



$$\underbrace{\Delta u}_{\text{polar form}} = c^2 \frac{\partial^2 u}{\partial t^2}$$

Try $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$

Get 3 ODE Probs:

1) $\Theta'' + \lambda \Theta = 0$

$$\left.\begin{aligned} \Theta(0) &= \Theta(2\pi) \\ \Theta'(0) &= \Theta'(2\pi) \end{aligned}\right\} \text{periodic B.C.}$$

2) $r^2 R'' + r R' + (r^2 - \lambda) R = 0$

Bessel's Eqn.

B.C. $R(1)=0$

$\hookrightarrow$ R bounded near $r=0$.

singular boundary cond. $P=\frac{1}{r}$, $Q=\frac{r^2-\lambda}{r^2}$

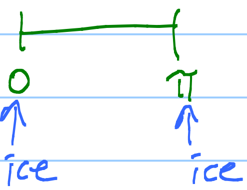
Solve by power series. Use BC.

Get Bessel fcns $J_n(x)$.

They form a complete $\perp$ system!

3) $T''+\lambda T=0 \leftarrow$ easy

<u>Heat prob:</u> Hot wire



I.C. $u(x,0)\equiv 1$

B.C. $\begin{cases} u(0,t)=0 \\ u(\pi,t)=0 \end{cases}$

PDE: $\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$ Try $u(x,t)=X(x)T(t)$

$$\begin{cases} X''-\lambda X=0 \\ X(0)=0,\ X(\pi)=0 \end{cases}$$

$\lambda=-n^2$: $X_n(x)=\sin nx$

$T'-\lambda T=0$

$T=ce^{\lambda t}$

Take $c=1$.

$T_n(t)=e^{-n^2 t}$

$$u(x,t)=\sum_{n=1}^{\infty} c_n \sin nx\, e^{-n^2 t}$$

Last step. IC: $u(x,0)=\underbrace{\sum_{n=1}^{\infty} c_n \sin nx}_{\text{F. Sine Series!}} \overset{\text{want}}{=} 1$

$$c_n=\frac{2}{\pi}\int_0^{\pi} 1\cdot \sin nx\,dx=\frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi}$$

$$=\frac{2}{\pi n}\Big[1-\underbrace{\cos n\pi}_{\substack{=-1 \ n \text{ odd} \\ =1 \ n \text{ even}}}\Big]=\begin{cases} \frac{4}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$u(x,t)=\frac{4}{\pi}\left(e^{-t}\sin x+\frac{1}{3}e^{-3^2 t}\sin 3x+\frac{1}{5}e^{-5^2 t}\sin 5x+\cdots\right)$$