Lesson 3 7.4

HWK 1 : Lessons 1,2,3 due Wed. 9/2, 11:59 pm Blackboard

LastnameFirstname_1.pdf

Office Hours: Tues., Wed. 2:00-3:00 pm in MATH 750.

765-494-1497

Next week: Tues., 8-9 pm WebEx

Wed., 8-9 pm Piazza

EX: $\mathbb{A}\vec{x} = \vec{b}$ $\quad \underbrace{[\mathbb{A} \mid \vec{b}\,]}_{\mathbb{A}^\#} \rightsquigarrow \left[\underbrace{\begin{matrix} \textcircled{5} & -2 & 1 & -3 \\ 0 & 0 & \textcircled{3} & 4 \\ 0 & 0 & 0 & 0 \end{matrix}}_{\mathbb{E}} \;\middle|\; \begin{matrix} 6 \\ 7 \\ 8 \end{matrix}\right]$

$\left.\begin{array}{l} \text{Rank}(\mathbb{A}) = 2 \\ \text{Rank}(\mathbb{A}^\#) = 3 \end{array}\right\}$ ouch!

No solutions

Replace 8 by 0: x_1, x_3 bound vars.

x_2, x_4 free vars. ← ∞ many sol^{n}s.

Def^n: $\text{Null}(\mathbb{A})$ = "null space of $\mathbb{A}$" = $\{\vec{x} : \mathbb{A}\vec{x} = \vec{0}\}$

$\text{Rank}(\mathbb{A})$ = # bound vars = # non-zero rows in $\mathbb{E}$.

$\text{Nullity}(\mathbb{A})$ = # free vars = "Dimension of the null space"

Fact: (# bound) + (# free) = # vars

$\text{Rank}(\mathbb{A}) + \text{Nullity}(\mathbb{A})$ = # columns of $\mathbb{A}$

$\mathbb{R}^n$ is a the original "vector space."

$\text{Null}(\mathbb{A})$ is a vect. sp. too.

Subspace Test: If V is a vector space, we want to know when $W \subset V$ also satisfies vect. sp. laws.

Fact: Only need to check:

1) If $\vec{w}_1$ and $\vec{w}_2$ are in W, so is $\vec{w}_1 + \vec{w}_2$.

(closed under addition.)

2) If $\vec{w}$ is in W, then so is $c\vec{w}$ for all c.

(closed under scalar mult.)

Check: $W = \text{Null}(\mathbb{A})$. 1) $\vec{x}_1, \vec{x}_2$ in $\text{Null}(\mathbb{A})$: $\mathbb{A}\vec{x}_1 = \vec{0}$

$\mathbb{A}\vec{x}_2 = \vec{0}$

Hmmm: $\mathbb{A}(\vec{x}_1 + \vec{x}_2) = \mathbb{A}\vec{x}_1 + \mathbb{A}\vec{x}_2 = \vec{0} + \vec{0} = \vec{0}$ ✓

2) $\vec{x}$ in Null(A). $A\vec{x} = \vec{0}$

$$A(c\vec{x}) = cA\vec{x} = c\vec{0} = \vec{0} \checkmark$$

EX: $W = \left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \in \mathbb{R}^3, \text{ a, b any } \mathbb{R} \right\}$

1. easy
2. easy ✓

EX: $\left\{ \begin{pmatrix} a \\ b \\ 3 \end{pmatrix} \in \mathbb{R}^3 \right\}$.

1. $\begin{pmatrix} a_1 \\ b_1 \\ 3 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ 3 \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \\ 6 \end{pmatrix}$ ouch!

or 2. $7\begin{pmatrix} a \\ b \\ 3 \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \\ 21 \end{pmatrix}$ ← ouch!

Basis and Dimension: $\mathbb{R}^3 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\underset{\vec{e}_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} + y\underset{\vec{e}_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} + z\underset{\vec{e}_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\}$

$\vec{e}_1, \vec{e}_2, \vec{e}_3$ "span" $\mathbb{R}^3$. They are independent. They form a basis for $\mathbb{R}^3$.

$\mathbb{R}^3$ is 3-dim^{e}.

Def^{n}: $\text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m) = \{ \underbrace{c_1\vec{v}_1 + \cdots + c_m\vec{v}_m}_{\text{linear combo}} : c\text{'s real} \}$

Fact: The span is a vector subspace of the $\mathbb{R}^n$ we live in. (Subspace Test. ✓)

Def^{n}: $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent if:

Whenever $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$,

all the c's must be zero.

Question: Are $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$ lin. indep.?

Suppose $c_1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + c_3\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}}_{A} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \vec{0}$$

$$A \rightsquigarrow \begin{bmatrix} \enclose{circle}{1} & 0 & -1 \\ 0 & \enclose{circle}{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

↑ c_3 free var.
∞ many sol^{n}s.

Can have non-zero c's.

They are not indep. They are dependent.

Let $c_3=1$. $\quad c_1\vec{v}_1+c_2\vec{v}_2+1\vec{v}_3=0$

$$\vec{v}_3=-c_1\vec{v}-c_2\vec{v}_2.$$

$\vec{v}_3$ is a lin combo of others.

Better way: How to decide if $\vec{v}_1,\dots,\vec{v}_n$ are indep and find a basis for the span.

Step 1: Put $\vec{v}$'s in as the rows of a matrix A.

Step 2: Row reduce A to row echelon E.

Step 3: If the bottom row is not all zeroes, they are indep. (If it is all 0s, they are dep.)

Bonus: The non-zero rows of E form a basis for the span (after you turn them back into column vects.)

(Basis = Lin. indep. set of spanning vectors.)

EX: Are $\begin{pmatrix}1\\4\\7\end{pmatrix}, \begin{pmatrix}2\\5\\8\end{pmatrix}, \begin{pmatrix}3\\6\\9\end{pmatrix}$ indep?

1. $\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}=A$

2. $A \rightsquigarrow \begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\end{bmatrix}$ $\leftarrow$ darn!

Not indep. They are dep.

Bonus: $\operatorname{Span}\left(\begin{pmatrix}1\\4\\7\end{pmatrix}, \begin{pmatrix}2\\5\\8\end{pmatrix}, \begin{pmatrix}3\\6\\9\end{pmatrix}\right)=\operatorname{Span}\left(\underbrace{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}}_{\text{Basis!}}\right)$

The span is 2-dim^l.

Why it works: Row operations do not change the span of the row vectors.

Great fact: $\operatorname{Rank}(A)=\operatorname{Rank}(A^T)$

Consequently, can stick $\vec{v}_1, \dots, \vec{v}_n$ in as <u>columns</u> of a matrix B. If $\text{Rank}(B) = n$, then they are <u>indep.</u> . If $\text{Rank}(B) < n$, they are <u>dep</u>.

And $\{\vec{v}_j : x_j$ is a <u>bound</u> var in $B\vec{x} = 0\}$

is a basis for span.

$$[\vec{v}_1, \sim, \vec{v}_n] \rightsquigarrow \begin{bmatrix} \textcircled{1} & 2 & 3 & 4 & 5 \\ 0 & 0 & \textcircled{4} & 5 & 6 \\ 0 & 0 & 0 & \textcircled{7} & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\vec{v}_1$, $\vec{v}_3$, $\vec{v}_4$ basis.