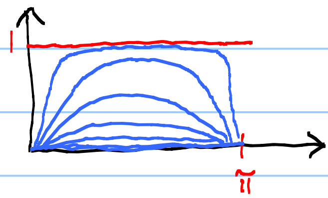
Lesson 30 on 11.7 Fourier Integral

26, 27, 28 due tonight
29, 30, 31 Wed. next
WebEx on Tuesdays 8-9 pm

Exam 2
Nov. 17

$$u(x,t) = \frac{4}{\pi}\left(e^{-t}\sin x + \frac{1}{3}e^{-3^2 t}\sin 3x + \frac{1}{5}e^{-5^2 t}\sin 5x + \cdots\right)$$

$\to 0$ fast



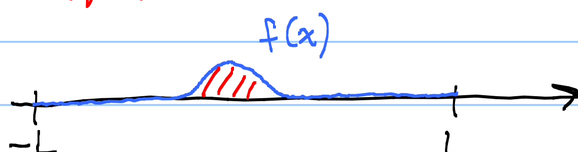
Besse''s Ineq: $\sum_{n=1}^{\cancel{N}\,\infty} c_n^2 \le \|f\|^2$

Conclude that $c_n \to 0$ as $n \to \infty$

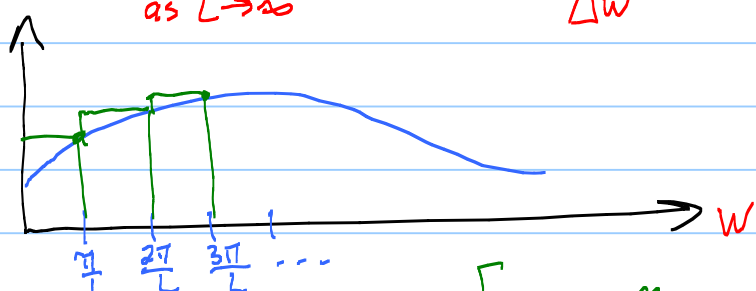
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\underbrace{\sin nx}_{\text{not}\to 0}\,dx \longrightarrow 0 \text{ as } n\to\infty$$

but wiggling cause cancellation in $\int$

Fourier Integral:



$$f(x) = \underbrace{\frac{1}{2L}\int_{-L}^{L} f(x)\,dx}_{\to 0 \text{ as } L\to\infty} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\left(\underbrace{\frac{\pi}{L}}_{\Delta w}\int_{-L}^{L} f(v)\cos \underbrace{\frac{n\pi v}{L}}_{w_n}\,dv\right)\cos\frac{n\pi}{L}x + \text{Similar Sine terms}\right]$$



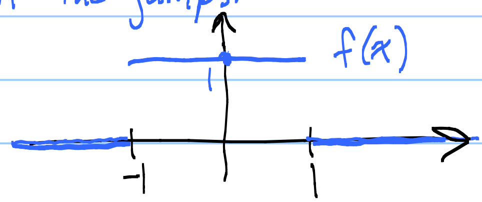
$$f(x) = \lim_{L\to\infty}\sum_{n=1}^{\infty}\left[\underbrace{\left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos w_n v\,dv\right)}_{A(w_n)}\cos w_n x\underbrace{\left(\frac{\pi}{L}\right)}_{\Delta w} + \text{Similar Sine}\right]$$

$$f(x) = \int_0^{\infty} A(w)\cos wx\,dw + \int_0^{\infty} B(w)\sin wx\,dw \quad \text{F.I.}$$

where
$$\begin{cases} A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv \\ B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv \end{cases}$$

Fact: If f is piecewise $\mathcal{C}^1$-smooth and $\int_{-\infty}^{\infty} |f(x)|\,dx$ is finite (i.e., f is absolutely integrable), then f is given by its F.I. But, at jumps, the F.I. converges to midpoints of the jumps.

EX:



$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{\underbrace{f(v)}_{\text{even}}\ \underbrace{\cos wv}_{\text{even}}}_{\text{even}}\,dv = \frac{2}{\pi}\int_0^{\infty} f(v)\cos wv\,dv$$

$$= \frac{2}{\pi}\int_0^1 1\cdot\cos wv\,dv = \frac{2}{\pi}\left[\frac{1}{w}\sin wv\right]_{v=0}^{1}$$

$$= \frac{2}{\pi}\left[\frac{1}{w}\sin w - \frac{1}{w}\underbrace{\sin 0}_{0}\right] = \frac{2}{\pi}\,\frac{\sin w}{w}$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{\underbrace{f(v)}_{\text{even}}\ \underbrace{\sin wv}_{\text{odd}}}_{\text{odd}}\,dv = 0$$

$$f(x) \text{ “=” } \int_0^{\infty} \underbrace{\frac{2}{\pi}\,\frac{\sin w}{w}}_{A(w)} \cos wx\,dw \;+\; \underset{\uparrow\ B(w)\equiv 0}{0}$$

$$\text{“=”} \begin{cases} 0 & x<-1 \\ 1/2 & x=-1 \\ 1 & -1<x<1 \\ 1/2 & x=1 \\ 0 & x>1 \end{cases}$$

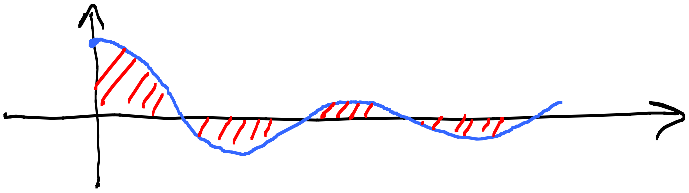
oops!

Interesting spot: $x=0$

$$f(0) = 1 = \int_0^{\infty} \frac{2}{\pi}\,\frac{\sin w}{w}\cdot 1\,dw$$

$\cos w\cdot 0 = \cos 0 = 1$

$$\int_0^{\infty} \frac{\sin w}{w}\,dw = \frac{\pi}{2}$$



Areas ↘ 0
Alt. Series Test ⇒ $\int_0^\infty$ converges

But $\int_0^\infty \left|\frac{\sin w}{w}\right| dw = \infty$ ← like $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

Facts: Odd f: $A(w) = 0$ ← no even stuff!

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} = \frac{2}{\pi}\underbrace{\int_0^\infty f(v)\sin wv\,dv}_{\text{Fourier Sine Integral}}$$

Even f:
$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} = \frac{2}{\pi}\underbrace{\int_0^\infty f(v)\cos wv\,dv}_{\text{Fourier Cosine Integral}}$$

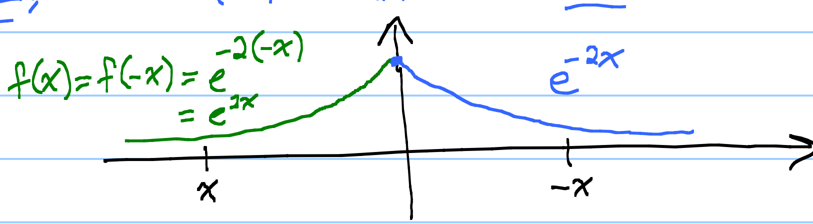
$B(w) = 0$ ← no odd stuff!

Prob: Expand $f(x) = e^{-2x}$ for $x > 0$ as a Fourier Cosine Integral.

↑ f can be anything on $x < 0$

↑ hint: extend f to be even.

Step 1: Extend f to $\mathbb{R}$ as an even fcn.



$$f(x) = \text{F.I.} = \underbrace{(\text{Cosine term})}_{\text{F. Cosine Int}} + (\text{no Sine term})$$

Step 2: Know $B(w) \equiv 0$.

$s = 2$ in $\mathcal{L}.T.$!

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} = \frac{2}{\pi}\int_0^\infty e^{-2v}\cos wv\,dv = \frac{2}{\pi}\,\frac{2}{2^2 + w^2}$$

Aha! $\mathcal{L}[\cos wt] = \int_0^\infty e^{-st} \cos wt \, dt = \frac{s}{s^2+w^2}$

So $e^{-2x} = \int_0^\infty \frac{4}{\pi(4+w^2)} \cos wx \, dw$

valid for $x > 0$.

503: 15. $y'' + 8y' + (\lambda+16)y = 0$ $\quad y(0)=0$, $y(\pi)=0$

$r^2 + 8r + (\lambda+16) = 0$

Roots $r = -4 \pm \sqrt{-\lambda}$

Case 1: $\lambda = 0$. $r = -4, -4$ double root

$y = c_1 e^{-4x} + c_2 x e^{-4x}$

$y(0) = c_1 \cdot 1 + c_2 \cdot 0 \overset{\text{want}}{=} 0 \qquad \underline{c_1 = 0}$

$y(\pi) = c_2 \pi e^{-4\pi} = 0 \qquad \underline{\underline{c_2 = 0}}$

$\lambda = 0$ not e-val.

Case 3: